North Dakota Mathematics Talent Search 2007-2008
Solutions Problem Set 3

1. Let a, b, c be odd integers. Prove that the quadratic equation $ax^2 + bx + c = 0$ does not have solutions in the set of rational numbers.

Solution: By contradiction, assume that the equation has a rational equation $x = m/n$, where m, n are relatively prime integers. Then

$$0 = a(m/n)^2 + b(m/n) + c = (1/n^2)(am^2 + bmn + cn^2),$$

which implies that $am^2 + bmn + cn^2 = 0$. However, since m and n are relatively prime, either m, n are both odd numbers or m, n have different parity. In both cases, since a, b, c are odd numbers, we have that $am^2 + bmn + cn^2$ is odd, hence it cannot be 0.

2. Prove that the product of eight consecutive positive integers cannot be a perfect square.

Solution: Let $x = (m-3)(m-2)(m-1)m(m+1)(m+2)(m+3)(m+4)$, where m is an integer with $m \geq 4$. Then

$$\begin{aligned} x &= (m^2 - 3m)(m^2 - 3m + 2)(m^2 + 5m + 4)(m^2 + 5m + 6) \\ &= [(m^2 - 3m + 1)^2 - 1][(m^2 + 5m + 5)^2 - 1] \end{aligned}$$

Set $a = m^2 - 3m + 1$ and $b = m^2 + 5m + 5$. Then

$$x = (a^2 - 1)(b^2 - 1) = (ab - 1)^2 - (a - b)^2 < (ab - 1)^2.$$

On the other hand, $x = (ab-1)^2-(a-b)^2 = (ab-2)^2+2(ab-2)+1-(a^2+b^2-2ab) = (ab-2)^2 + 4ab - a^2 - b^2 - 3$.

For $m \geq 8$, we will prove that $a^2+b^2 < 4ab-3$, which will imply that $x > (ab-2)^2$. Equivalently, we will show that $4a^2+b^2-4ab < 3(a^2-1)$, i.e., $(2a-b)^2 < 3(a^2-1)$.

To see this, it is enough to notice that $-(a-1) < 2a - b < a - 1$, which implies that $(2a-b)^2 < (a-1)^2$ and hence

$$(2a - b)^2 < (a - 1)^2 < 3(a - 1)(a + 1),$$

which is what we wanted. The inequality $2a-b < a-1$ is clear, and the inequality $-(a-1) < 2a - b$ is equivalent to $2m^2 - 14m - 3 > 0$, which is true for $m \geq 8$.

In conclusion, for $m \geq 8$ we proved that

$$(ab - 2)^2 < x < (ab - 1)^2,$$

so x is between two consecutive perfect squares, hence it cannot be a perfect square. For $m = 4, 5, 6, 7$, one can check directly that x is not a perfect square.

3. Let a, b, c, d be positive integers such that $ad = bc$. Prove that $a+b+c+d$ cannot be a prime number.

Solution: Since $\frac{a}{b} = \frac{c}{d}$, we can write $\frac{a}{b} = \frac{c}{d} = \frac{x}{y}$, where x and y are positive integers and $(x, y) = 1$ (x and y are relatively prime). Then there exist positive integers α and β such that $a = \alpha x, b = \alpha y, c = \beta x, d = \beta y$, and hence $a+b+c+d = \alpha x + \alpha y + \beta x + \beta y = (x + y)(\alpha + \beta)$, which is not prime.

4. Let n be an integer. If $n \geq 2$, prove that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ cannot be an integer.

Solution: Let k be the largest integer such that $2^k \leq n$. By contradiction, assume that the sum $S = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is an integer. Consider

$$2^{k-1}S = \frac{2^{k-1}}{2} + \frac{2^{k-1}}{3} + \cdots + \frac{2^{k-1}}{n}.$$

One of the fractions that appear in the above sum is $\frac{2^{k-1}}{2^k} = \frac{1}{2}$. Note that all the other fractions that appear in the sum can be simplified to a fraction with **odd** denominator (the largest power of 2 that can appear in the factorization of each of the numbers $2, 3, \ldots, 2^k - 1, 2^k + 1, \ldots, n$ is 2^{k-1}). So $2^{k-1}S = \frac{1}{2} + \frac{a}{b}$ where a and b are positive integers with b odd. On the other hand, by our assumption $\alpha = 2^{k-1}S$ is an integer, so $b + 2a = 2\alpha b$, which implies that b is even, a contradiction.

5. Find the positive integers n such that all the numbers $n + 1$, $n + 3$, $n + 7$, $n + 9$, $n + 13$, and $n + 15$ are prime.

Solution: If n is divisible by 5, then $n + 15$ is a multiple of 5 (greater than 5), hence it is not prime. If n gives remainder 1 when divided by 5, then $n + 9$ is a multiple of 5 (greater than 5), hence it is not prime. If n gives remainder 2 when divided by 5, then $n + 13$ is a multiple of 5 (greater than 5), hence it is not prime. If n gives remainder 3 when divided by 5, then $n + 7$ is a multiple of 5 (greater than 5), hence it is not prime. If n gives remainder 4 when divided by 5 and $n > 4$, then $n + 1$ is a multiple of 5 (greater than 5), hence it is not prime.

In the case $n = 4$, all the numbers $n + 1$, $n + 3$, $n + 7$, $n + 9$, $n + 13$, and $n + 15$ are prime, so $n = 4$ is the only positive integer that satisfies the given condition.