North Dakota Mathematics Talent Search 2007-2008
Solutions Problem Set 2

1. The student lockers at Nightmare Highschool are numbered consecutively beginning with locker number 1. The plastic labels used to number the lockers cost two cents apiece. Thus, it costs two cents to label locker number 9 and four cents to label locker number 11. If it costs \$137.94 to label all the lockers, how many lockers are there at the school?

Solution: We need 137.94/0.02=6897 digits to do the labeling. The first 9 lockers require 9 digits, lockers 10 through 99 require $2 \times 90 = 180$ digits, lockers 100 through 999 require $3 \times 900 = 2700$ digits. The remaining $6897 - 2700 - 180 - 9 = 4008$ digits are to be used for the rest of the lockers, each using 4 digits. So we have $4008/4 = 1002$ more lockers. The total number of lockers is therefore $1002 + 999 = 2001$.

2. Find the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$.

Solution: Let $x_i = 2y_i - 1$ $(i = 1, 2, 3, 4)$, where y_1, y_2, y_3, y_4 are positive integers. Then $98 = 2(y_1 + y_2 + y_3 + y_4) - 4$, so $y_1 + y_2 + y_3 + y_4 = 51$. Consider 51 balls in a row. Imagine that we split this row into 4 parts by inserting 3 dividers. This corresponds to a solution of the equation. The number of positive solutions of the equation $y_1 + y_2 + y_3 + y_4 = 51$ is equal to the number of ways one can split the row of balls into 4 parts by inserting 3 dividers. Since there are 50 spaces available for inserting the 3 dividers, we can do the splitting in $(50 \times 49 \times 48)/6 = 19600$ ways (we divided by $6 = 1 \times 2 \times 3$ because a permutation of the separators does not produce a different solution for the equation).

3. In how many ways can one arrange the numbers 21, 31, 41, 51, 61, 71, and 81 such that the sum of every four consecutive numbers is divisible by 3?

Solution: We only need to consider the remainders obtained when we divide our numbers by 3, that is $0, 1, 2, 0, 1, 2, 0$. We need to find the number of ways we can arrange these numbers such that the sum of every four consecutive numbers is divisible by 3. Let $a_1a_2 \ldots a_7$ be such an arrangement. Since $(a_1 + a_2 + a_3 + a_4) + (a_4 + a_5 + a_6 + a_7)$ is a multiple of 3 and $(a_1 + a_2 + \ldots + a_7) = 6$ is a multiple of 3, we must have that a_4 is a multiple of 3, so there are 3 possibilities for a_4. Since $a_1 + a_2 + a_3 + a_4$ is a multiple of 3, we must have that $a_1 + a_2 + a_3$ is a multiple of 3, which can only happen when a_1, a_2, a_3 are an arrangement consisting of 0,1, and 2. Therefore there are $2 \times 2 \times 2 \times 3! = 48$ ways we can choose the triplet $a_1a_2a_3$. On the other hand, let us notice that once a_1, a_2, a_3 are chosen, there is only one choice for the remaining numbers. Indeed, since $a_2 + a_3 + a_4 + a_5$ is a multiple of 3 and $a_1 + a_2 + a_3 + a_4$ is a multiple of 4, it follows that the difference $a_5 - a_1$ is a multiple of 3. This implies that $a_5 = a_1$. Similarly we must have $a_6 = a_2$ and $a_7 = a_3$. So the total number of ways we can arrange these numbers is $3 \times 48 = 144$.

4. We call a 7-digit telephone number $d_1d_2d_3$–$d_4d_5d_6d_7$ *memorable* if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). Assuming that each d_i can be any of the ten decimal digits $0, 1, \ldots, 9$, find the number of different memorable telephone numbers.

Solution: There are 1000 ways of choosing $d_1d_2d_3$. The number of memorable numbers with $d_4d_5d_6 = d_1d_2d_3$ is therefore $1000 \times 10 = 10000$. Similarly, the number of memorable numbers with $d_5d_6d_7 = d_1d_2d_3$ is also 10000. Also, a memorable number is counted twice only when $d_4d_5d_6 = d_5d_6d_7 = d_1d_2d_3$, that is $d_1 = d_2 = \ldots = d_7$. Therefore we counted twice 10 memorable numbers. So the total number of memorable numbers is $20000 - 10 = 19990$.

5. Nine chairs in a row are to be occupied by six students and Professors X, Y, and Z. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors X, Y, and Z choose their chairs?

Solution: Imagine only the 6 students standing up in a row. Each of the 5 spaces between them can be occupied by at most one professor. The professors can choose their positions in $5 \times 4 \times 3 = 60$ ways. Of course, at the end of this, tell everybody to sit down in the chosen order.