North Dakota Mathematics Talent Search 2007-2008
Solutions for Problem Set 1

1. In how many ways can we distribute 10 books to two persons?

Solution: Label the books 1 through 10. Let A and B be the two persons. One way to distribute the books is, for example, $ABABABABAB$, meaning that the person A receives the books labeled $1, 3, 5, 7, 9$ and person B receives the books labeled $2, 4, 6, 8, 10$. The number of ways can we distribute 10 books to two persons is therefore equal to the number of length 10 strings of As and Bs. This number is $2^{10} = 1024$.

2. Start writing down in increasing order the positive integers beginning with 1, say 123456789101112131415161718192021.... Find the 34788$^{\text{th}}$ digit in this chain.

Solution: There exist 9 numbers with one digit, 99-9=90 numbers with 2 digits, 999-99=900 numbers with 3 digits, 9999-999=9000 numbers with 4 digits. The number of digits needed to write the first 999 positive integers (beginning with 1) is $9 + 2 \times 90 + 3 \times 900 = 2889$, and the number of digits needed to write the first 9999 positive integers is $2889 + 4 \times 9000 = 38889$. So the number that contains the 34788$^{\text{th}}$ digit must have 4 digits.

The number of digits needed to write the 4-digit numbers up to (and including) the 34788$^{\text{th}}$ digit is $34788 - 2889 = 31899$. As each number has 4 digits and $31899 = 4 \times 7974 + 3$, it follows that the number that contains the 34788$^{\text{th}}$ digit is the 7975$^{\text{th}}$ number with 4 digits, that is $7975 + 999 = 8974$. The digit we are looking for is the third digit of this number, namely 7.

3. Consider an $m \times n$ table with m rows and n columns whose entries consist of the integers $+1$ and -1 such that the product of the elements on each row and on each column is -1.

(a) Give an example of such a table for $m = 3$ and $n = 5$.

(b) Show that if such an $m \times n$ table exists, then m and n have the same parity.

Solution: A 3×5 table with all entries -1 is such an example.

In the table there are mn entries. We will compute the product p of all these entries in two ways. On one hand, since the product of the entries on each line is -1 we have $p = (-1)^m$. On the other hand, since the product of the entries on each column is also -1, we must also have $p = (-1)^n$. So $(-1)^m = (-1)^n$, which implies that m and n have the same parity.

4. How many positive integer solutions does the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$ have ? (For $x \neq y$ the solutions (x, y) and (y, x) are considered different.)

Solution: Since $\frac{1}{x} = \frac{1}{12} - \frac{1}{y} < \frac{1}{12}$, we must have $x > 12$. Similarly, we must also have $y > 12$. The equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$ can be written $12x + 12y = xy$, or equivalently, $(x-12)(y-12) = 144 = 2^4 \times 3^2$. The number of solutions is therefore equal to the number of positive divisors of $2^4 \times 3^2$, which is $(4+1)(2+1) = 15$.

5. (a) Find the exponent of 2 in the prime factor decomposition of $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 99 \cdot 100$.

(b) How many ending zeros does $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 99 \cdot 100$ have ?

Solution: Among the numbers $1, 2, \ldots, 100$ there are 50 numbers that are multiple of 2. Among these 50 numbers, 25 of them are multiple of $2^2 = 4$. Among these 25 numbers, 12 of them are divisible by $2^3 = 8$. Among these 12 numbers, 6 of them are divisible by $2^4 = 16$. Among these 6 numbers, 3 of them are divisible by $2^5 = 32$. Among these 3 numbers, only one is divisible by $2^6 = 64$. There are no numbers divisible by $2^7 = 128$. So the exponent of 2 from the prime factor decomposition of $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 99 \cdot 100$ is $50 + 25 + 12 + 6 + 3 + 1 = 97$.

Using the same argument as above, the exponent of 5 from the prime factor decomposition of $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 99 \cdot 100$ is $20 + 4 = 24$. Since $10 = 2 \times 5$, the number of ending zeros of $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 99 \cdot 100$ is 24.