North Dakota Mathematics Talent Search 2006-2007
Problem Set 3
Problems due May 15^{th}, 2007

1. Let a, b, c be real numbers such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}.$$

Prove that

$$\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=\frac{1}{a^5+b^5+c^5}.$$

Solution: From $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$. we obtain $(a+b+c)(ab+bc+ca)-abc=0$. Equivalently, $(a+b)(ab+bc+ca+c^2)-(a+b)c^2+c(ab+bc+ca)-abc=0$, i.e, $(a+b)(ab+bc+ca+c^2)=0$, $(a+b)(b+c)(c+a)=0$. From here we obtain that at least one of the following equalites occurs: $a=-b$, $b=-c$, $c=-a$. In any case, it follows that $\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=\frac{1}{a^5+b^5+c^5}$.

2. Find $\max\{2^{237}, 3^{158}\}$.

Solution: Since $8=2^3<3^2=9$, we obtain $2^{237}=(2^3)^{79}<(3^2)^{79}=3^{158}$.

3. The arithmetic average of n numbers $a_1, a_2, \ldots, a_n$ is given by

$$\frac{a_1+a_2+\ldots+a_n}{n}.$$

Assume that the arithmetic average of 50 numbers is 10. If the arithmetic average of the first 40 numbers is 12, find the arithmetic average of the last 10 numbers.

Solution: Let $a_1, a_2, \ldots, a_{50}$ be the 50 numbers. We know that $a_1+\cdots+a_{50}=500$ and $a_1+\cdots+a_{40}=480$, and hence $a_{41}+\cdots+a_{50}=20$, which means that the arithmetic average of the last 10 numbers is 2.

4. Assume that a and b are positive integers such that each of them can be written as a sum of two perfect squares. Show that ab can also be written as a sum of two perfect squares.

Solution: Write $a=x^2+y^2$ and $y=z^2+w^2$ with x, y, z, w integers. Then $ab=(x^2+y^2)(z^2+w^2)=x^2z^2+x^2w^2+y^2z^2+y^2w^2=(x^2z^2+y^2w^2+2xyzw)+(y^2z^2+x^2w^2-2xyzw)=(xz+yw)^2+(yz-xw)^2$ which is also a sum of two perfect squares.

5. Let A be a set consisting of five integers. Show that A contains two distinct elements a and b such that either $a+b$ or $a-b$ is a multiple of 7.

Solution: An integer x is of one of the following forms: $7k$, $7k+1$, $7k-1$, $7k+2$, $7k-2$, $7k+3$, $7k-3$ (k integer). The square of an integer is of one of the following forms: $7s, 7s + 1, 7s + 4, 7s + 2$. In our problem we are given five integers, say a, b, c, d, e. This means that at least two of the integers a^2, b^2, c^2, d^2, e^2 give the same ramainder when divided by 7. Assume that these two integers are a^2 and b^2. This means that $a^2 - b^2 = (a + b)(a - b)$ is divisible by 7, so either $a + b$ or $a - b$ is a multiple of 7.