North Dakota Mathematics Talent Search 2006-2007
Solutions for Problem Set 2

1. Of 100 people seated at a round table, more than half are children. Prove that there are two children who are seating diametrically opposite each other.

Solution: There are 50 pairs of people who are seating diametrically opposite each other. Since there are more than 50 children, at least one pair must contain more than one child.

2. Show that in any group of five people, there are two who have an identical number of friends within the group.

Solution: The number of acquaintances for each person is 0, 1, 2, 3 or 4. If no two persons have the same number of acquaintances, then we have a person A who has 0 acquaintances and a person B with 4 acquaintances. But then B must be a friend of A, contradicting that A has no acquaintances.

3. Twenty-five boys and 25 girls are seated at a round table. Show that both neighbors of at least one student are boys.

Solution: Assume that no student has two neighbors who are boys. Count the positions at the table starting at some point and going around the table (do the counting modulo 50).

If there is a boy in the k-th position, then there is a girl either in position $k-1$ or in position $k+1$. If position $k+1$ is a girl, then $k+2$ cannot be a boy. If position $k+1$ is a boy, then a girl must be in position $k+2$. Similarly, one can show that a girl must be in position $k-2$.

With a similar argument, if there is a girl in the k-th position, then there are boys on both $k-2$ and $k+2$ positions.

We now consider the group of 25 students who are seating in even places. By the above arguments, we find that the boys and the girls alternate around the table (in this group). But this is impossible since 25 is an odd number.

4. Given the pair of prime numbers p and $8p^2+1$, find p.

Solution: If p gives remainder 1 when divided by 3 ($p = 3k+1$), then $8p^2 + 1 = 8(3k+1)^2 + 1 = 72k^2 + 48k + 9$ is a multiple of 3 (greater than 9), so it cannot be a prime. If p gives remainder 2 when divided by 3 ($p = 3k+2$), then $8p^2 + 1 = 8(3k+2)^2 + 1 = 72k^2 + 96k + 33$ is also a multiple of 3 (greater than 33), so it cannot be a prime.

The only case left is when p is a multiple of 3. But p is a prime, so we must have $p = 3$. In this case $8p^2 + 1 = 73$ which is a prime.

5. Consider the set $A = \{a_1, a_2, \ldots, a_n\}$ consisting of integers. Prove that A has a non-empty subset with the property that the sum of its elements is a multiple of n.

Solution: Consider the following n subsets of A:

$$A_1 = \{1\}, A_2 = \{a_1, a_2\}, A_3 = \{a_1, a_2, a_3\}, \ldots, A_n = \{a_1, a_2, \ldots, a_n\}.$$

The sum of the elements for each of these subsets is:

$S_1 = a_1, S_2 = a_1 + a_2, \ldots, S_n = a_1 + a_2 + \ldots + a_n$.

If one of the numbers $S_1, S_2, \ldots, S_n$ is divisible by n, the problem is solved.

If not, the n remainders obtained by dividing these numbers by n belong to the set $\{1, 2, \ldots, n-1\}$. So there exist two numbers S_i and S_j that give the same remainder when divided by n. Let

$$S_i = a_1 + a_2 + \ldots + a_i \quad \text{and } S_j = a_1 + a_2 + \ldots + a_j, \text{ where } i < j.$$

Then n divides the difference $S_j - S_i = a_{i+1} + a_{i+2} + \ldots + a_j$, so $\{a_{i+1}, a_{i+2}, \ldots, a_j\}$ is a subset of A with the property that the sum of its elements is a multiple of n.