North Dakota Mathematics Talent Search 2006-2007
Solutions for Problem Set 1

1. Find the last digit of 1989^{1989} (in decimal notation).

Solution: The last digit of 1989^{1989} is the same as the last digit of 9^{1989}. Notice that for every positive integer k, the last digit of $9^{2k} = 81^k$ is 1 and the last digit of $9^{2k+1} = 9 \cdot 81^k$ is 9. In particular, the last digit of 1989^{1989} is 9.

2. Let $a_1, a_2, \ldots, a_{11}$ be eleven distinct positive integers, each less than 21. Prove that among these numbers one can find two numbers such that one of them divides the other one.

Solution: Let $X = \{1, 2, \ldots, 20\}$ and $A = \{11, 13, 15, 17, 19\}$. Since A has 5 elements, at least 6 of the integers $a_1, \ldots, a_{11}$ are from $B := X \setminus A$.

Let $B_1 = \{1, 2, 4, 8, 16\}$, $B_2 = \{3, 6, 12\}$, $B_3 = \{5, 10, 15\}$, $B_4 = \{7, 14\}$ and $B_5 = \{9, 18\}$. Notice that $B_1, B_2, \ldots, B_5$ are 5 disjoint sets forming a partition of B. Since at least 6 of the integers $a_1, \ldots, a_{11}$ are from B, it follows that at least two of them are from the same B_j for some j. On the other hand, if we choose a pair of numbers from B_j, one must divide the other one.

3. Can a 8×8 chessboard be covered with 1×2 dominoes such that only $a1$ and $h8$ remain uncovered?

Solution: No. Each domino covers one black square and one white square, but if we leave out the squares $a1$ and $h8$, there will be two more white squares than black squares remaining.

4. A group of 40 students participated in a math contest that consists of four problems. Out of the 40 students, 25 solved the first problem, 30 students solved the second problem, 35 solved the third problem and 33 solved the fourth problem. Prove that at least 3 students solved all four problems.

Solution: Assume that at most two students solved all of the four problems. The number of solutions given by students for the first 3 problems is at most $40 \cdot 3 = 120$, hence the total number of solutions for all four problems is at most $120 + 2 = 122$.

On the other hand, we know that the total number of solutions is $25+30+35+33 = 123$, contradiction. This implies that at least 3 students solved all four problems.

5. Let $\{a_1, a_2, \ldots, a_9\}$ be a permutation of the numbers from 1 to 9. Show that the product

$$(a_1 - 1)(a_2 - 2) \ldots (a_9 - 9)$$

is an even number.

Solution: Assume that the product $(a_1 - 1)(a_2 - 2) \ldots (a_9 - 9)$ is odd. Then all the numbers $(a_1 - 1), (a_2 - 2), \ldots, (a_9 - 9)$ are odd, and hence a_1, a_3, a_5, a_7, a_9 are even and a_2, a_4, a_6, a_8 are odd. On the other hand, since $\{a_1, a_2, \ldots, a_9\}$ are a permutation of the numbers from 1 to 9, only 4 of them are even and 5 of them are odd, contradiction.

6. The sum of the digits of a positive integer written in decimal notation is 2006. Can this number be a perfect square?

Solution: The remainder obtained by dividing a positive integer by 3 is equal to the remainder obtained by dividing the sum of its digits by 3. Indeed,

$$\overline{a_k a_{k-1} \ldots a_1} - (a_k + a_{k-1} + \ldots + a_1) = (10^{k-1} - 1)a_k + (10^{k-2} - 1)a_{k-1} + \ldots + (10^1 - 1)a_2$$

which is divisible by 3.

Since the sum of the digits of our positive integer is 2006, it follows that the remainder obtained by dividing our integer by 3 is 2. On the other hand, since every positive integer is of one of the forms $3k, 3k+1, 3k-1$ (k integer), it follows that the square of a positive integer is of one of the forms $3p, 3p+1$ (p integer), and hence there are no perfect squares whose remainder obtained by dividing by 3 is 2.

7. Prove that the distance between any two points inside a triangle is not greater than half the perimeter of the triangle.

Solution: If the two points are denoted X, Y, extend the segment connecting them until it intersects the sides of the triangle in E and F. Assume that X is between E and Y. Also, denote the vertices of the triangle by A, B, C such that E is on the segment line AB and F is on the segment line AC. Then $EF < EA + AF$ and $EF < EB + BC + CF$. Adding, we obtain that $EF < (AB + BC + CA)/2$. Since $XY < EF$, XY is also less than half the perimeter of the triangle.