North Dakota Mathematics Talent Search 2005-2006
Problem Set 3 – Solutions

1. Find two disjoint subsets A and B of the set of positive rationals $\mathbb{Q}_+^*$ such that $A \cup B = \mathbb{Q}_+^*$ and $B = \{xy \mid x, y \in A\}$.

Solution: For example, let

$$A := \{x \in \mathbb{Q}_+^* \mid x = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}, k \in \mathbb{N}, p_1, \dots, p_k \text{ primes}, \\ \alpha_1, \alpha_2 \dots \alpha_k \in \mathbb{Z}, \alpha_1 + \alpha_2 + \dots + \alpha_k \text{ odd}\}$$

and

$$B := \{x \in \mathbb{Q}_+^* \mid x = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}, k \in \mathbb{N}, p_1, \dots, p_k \text{ primes}, \\ \alpha_1, \alpha_2 \dots \alpha_k \in \mathbb{Z}, \alpha_1 + \alpha_2 + \dots + \alpha_k \text{ even}\}.$$

Note that $B = \mathbb{Q}_+^* \setminus X$.

We also have $B = \{xy \mid x, y \in A\}$. Indeed, if $\alpha_1 + \alpha_2 + \dots + \alpha_k$ is even, then $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = p_1 \cdot (p_1^{\alpha_1 - 1} p_2^{\alpha_2} \dots p_k^{\alpha_k})$ and the inclusion $\subseteq$ follows. The other inclusion is immediate.

2. For each integer $n \geq 2$, denote by $p(n)$ the greatest prime number less than or equal to n, and by $q(n)$ the smallest prime number strictly greater than n.

(a) Prove that $S(n) = \displaystyle\sum_{k=2}^{n} \frac{1}{p(k)q(k)} < \frac{1}{2}$.

(b) Prove that $\dfrac{2}{1 - 2S(2010)}$ is an integer.

Solution: (a) Let r_k denote the k-th prime number ($r_1 = 2$). Note that for $r_i \leq j < r_{i+1}$ we have $p(j) = r_i$ and $q(j) = r_{i+1}$. Then we have

$$S(n) = \sum_{k=2}^{n} \frac{1}{p(k)q(k)} \tag{1}$$

$$= (r_2 - r_1)\frac{1}{r_1 r_2} + (r_3 - r_2)\frac{1}{r_2 r_3} + \dots + (n + 1 - r_t)\frac{1}{r_t r_{t+1}} \tag{2}$$

where $r_t \leq n < r_{t+1}$.

On the other hand, the sum in (**??**) is bounded above by

$$(r_2 - r_1)\frac{1}{r_1 r_2} + (r_3 - r_2)\frac{1}{r_2 r_3} + \dots + (r_{t+1} - r_t)\frac{1}{r_t r_{t+1}}$$
$$= \frac{1}{r_1} - \frac{1}{r_{t+1}} = \frac{1}{2} - \frac{1}{r_{t+1}}$$

and the conclusion follows.

(b) Since 2011 is prime, we have $r_{t+1} = 2011$ and from (**??**) we obtain

$$\begin{aligned} S(2010) &= (r_2 - r_1)\frac{1}{r_1 r_2} + (r_3 - r_2)\frac{1}{r_2 r_3} + \ldots + (2011 - r_t)\frac{1}{r_t \cdot 2011} \\ &= \frac{1}{2} - \frac{1}{2011}, \end{aligned}$$

and hence $\dfrac{2}{1 - 2S(2010)} = 2011$.

3. Find all the positive integers n with the property that there exist integers a, b with $n^2 = a + b$ and $n^3 = a^2 + b^2$.

Solution: We have $2n^3 = 2a^2 + 2b^2 \geq (a + b)^2 = n^4$, and hence $n \leq 2$. For $n = 1$ note that we have $1 + 0 = 1^2$ and $1^2 + 0^2 = 1^3$, and for $n = 2$ we have $2 + 2 = 2^2$ and $2^2 + 2^2 = 2^3$. So both $n = 1$ and $n = 2$ are solutions to the problem.