# North Dakota Mathematics Talent Search 2005-2006
## Problem Set 2–Solutions

**1.** Find the remainder obtained when we divide $N = 6^{102} + 6^{103} + \ldots + 6^{2006}$ by 43.

**Solution:** Note that $N = 6^{102} + 6^{103} + \ldots + 6^{2006} = 6^{102}(1 + 6 + 6^2) + 6^{105}(1 + 6 + 6^2) + \ldots + 6^{2004}(1 + 6 + 6^2) = (6^{102} + 6^{105} + \ldots + 6^{2004}) \cdot 43$, and hence $N$ is divisible by 43, i.e., the remainder is 0.

**2.** Let $x$ be a real number such that $x + \dfrac{1}{x}$ is an integer. Prove that $x^5 + \dfrac{1}{x^5}$ is also an integer.

**Solution:** Let $z = x+\dfrac{1}{x} \in \mathbb{Z}$. Then $z^2 = x^2+\dfrac{1}{x^2}+2$ and $z^3 = x^3+\dfrac{1}{x^3}+3\Big(x+\dfrac{1}{x}\Big)$, and hence $(z^2-2)(z^3-3z) = \Big(x^2+\dfrac{1}{x^2}\Big)\Big(x^3+\dfrac{1}{x^3}\Big) = x^5+\dfrac{1}{x^5}+x+\dfrac{1}{x}$, or equivalently, $x^5 + \dfrac{1}{x^5} = (z^2 - 2)(z^3 - 3z) - z$, which is an integer.

**3.** Consider the polynomial $P(X) = 3X^2 + 3mX + m^2 - 1$, where $m$ is a fixed real number, and let $a, b$ be the solutions of the equation $P(X) = 0$. Prove that $P(a^3) = P(b^3)$.

**Solution:** $P(a^3) - P(b^3) = 3(a^6 - b^6) + 3m(a^3 - b^3) = 3(a^3 - b^3)(a^3 + b^3 + m)$. On the other hand, since $a + b = -m$ and $ab = (m^2 - 1)/3$, we have $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = -m^3 + m(m^2 - 1) = -m$, and therefore $P(a^3) - P(b^3) = 0$.

**4.** Let $n$ be an integer greater than 1. Prove that $3^n+5^n$ is not divisible by $3^{n-1}+5^{n-1}$.

**Solution:** Note that for $n > 1$ we have $3^n + 5^n = 4(3^{n-1} + 5^{n-1}) + (5^{n-1} - 3^{n-1}) > 4(3^{n-1}+5^{n-1})$. Similarly, for $n > 1$, we also have $3^n+5^n = 5(3^{n-1}+5^{n-1})-2\cdot 5^{n-1} < 5(3^{n-1} + 5^{n-1})$. Hence, for $n > 1$,

$$4 < \frac{3^n + 5^n}{3^{n-1} + 5^{n-1}} < 5,$$

and therefore $\dfrac{3^n + 5^n}{3^{n-1} + 5^{n-1}}$ cannot be an integer.

**5.** Let $S = \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{2006^2}$. Prove that $S < \dfrac{2005}{2006}$.

**Solution:** For every $n \geq 2$ we have $\dfrac{1}{n^2} < \dfrac{1}{n(n-1)} = \dfrac{1}{n-1} - \dfrac{1}{n}$, and hence

$$\begin{aligned} S = \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{2006^2} &< \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \ldots + \Big(\frac{1}{2005} - \frac{1}{2006}\Big) \\ &= 1 - \frac{1}{2006} = \frac{2005}{2006}. \end{aligned}$$

Send your solutions to:

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Please include:

Name:
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You may also submit your solutions by e-mail to catalin.ciuperca@ndsu.edu.