**North Dakota Mathematics Talent Search 2005-2006**
**Problem Set 1**
**Solutions**

**1.** Given five points inside a square of size 1x1, prove that at least two of them are situated at a distance less than 0.75 from each other.

**Solution:** Divide the square into four $(1/2) \times (1/2)$ squares. Since we have five points and four squares, two of the points will be inside the same square. The distance between those two points will be at most $\sqrt{2}/2$ (the length of the diagonal of a $(1/2) \times (1/2)$ square), and hence less than 0.75.

**2.** (a) The sum of $2n$ positive integers is $2n^2 + n - 1$. Prove that at least two of these integers are equal.

(b) The sum of $2n$ positive integers is $n^2 + n - 1$. Prove that at least three of these integers are equal.

**Solution:** (a) Let $a_1 \leq a_2 \leq \ldots \leq a_{2n}$ be the $2n$ positive integers. By contradiction, assume that these integers are distinct. Then $a_1 < a_2 < \ldots < a_{2n}$ and therefore $a_1 \geq 1$, $a_2 \geq 2$, $\ldots$, $a_{2n} \geq 2n$. But then $a_1 + \ldots + a_{2n} \geq 1 + 2 + \ldots + 2n = 2n^2 + n$, contradiction.

(b) Let $a_1 \leq a_2 \leq \ldots \leq a_{2n}$ be the $2n$ positive integers and let $s_1 = a_1 + a_3 + \ldots + a_{2n-1}$ and $s_2 = a_2 + a_4 + \ldots + a_{2n}$. Note that $s_1 \leq s_2$, and since $s_1 + s_2 = n^2 + n - 1$, we also have $s_1 \leq (n^2 + n - 1)/2$. We claim that at least two of the integers $a_1, a_3, \ldots, a_{2n-1}$ are equal. Indeed, if not, then $a_1 < a_3 < \ldots < a_{2n-1}$ and hence $a_1 \geq 1, a_3 \geq 2, \ldots, a_{2n-1} \geq n$. But then $s_1 \geq 1 + 2 + \ldots + n = (n^2 + n)/2$, contradiction. So there exists $i$ odd such that $a_i = a_{i+2}$. Since $a_i \leq a_{i+1} \leq a_{i+2}$, we get $a_i = a_{i+1} = a_{i+2}$, q.e.d.

**3.** A group of dogs and cats went to a pizza restaurant where all the pizzas had exactly 12 slices. Each dog ate 6 or 7 slices and each cat ate 2 or 3 slices. Four pizzas were not enough for them, but they could not finish the fifth one. How many dogs and cats went to the restaurant?

**Solution:** Denote by $x, y$ the number of dogs and cats, respectively. Also, let $N$ be the total number of slices of pizza that were eaten. Since four pizzas were not enough but they could not finish the fifth one, we must have $49 \leq N \leq 59$. Since each dog ate at least 6 slices and each cat ate at least 2 slices, we get $N \geq 6x + 2y$. Similarly, since each dog ate at most 7 slices and each cat at most 3 slices, we also have $7x + 3y \geq N$. Hence we must have

$$6x + 2y \leq 59$$

and

$$7x + 3y \geq 49.$$

Note that the first inequality implies that $x \leq 9$. So the pairs $(x, y)$ that satisfy the problem are:

$$\{(x, y) \mid x, y \text{ positive integers}, 1 \leq x \leq 9, (49 - 7x)/3 \leq y \leq (59 - 6x)/2\}.$$

For example, in the case $x = 7$ and $y = 8$, if each dog ate only 6 slices, and each cat only two slices, they would have eaten a total of $7 \times 6 + 8 \times 2 = 58$ slices.

**4.** Let $a, b, c$ be distinct nonzero real numbers. Assume that the quadratic equations $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ have a common root. Prove that the remaining roots satisfy the quadratic equation $x^2 + cx + ab = 0$.

**Solution:** Let $\beta, \gamma$ be the roots of $x^2 + ax + bc = 0$ and let $\gamma, \alpha$ be the roots of $x^2 + bx + ca = 0$. Then $\gamma^2 + a\gamma + bc = 0$ and $\gamma^2 + b\gamma + ca = 0$, which implies that $\gamma^2 + a\gamma + bc - \gamma^2 - b\gamma - ca = 0$, i.e., $(\gamma - c)(a - b) = 0$. Since $a, b, c$ are distinct real numbers, we get $\gamma = c$. But $\gamma$ is a root of $x^2 + ax + bc = 0$ , so $c^2 + ac + bc = 0$, and therefore $a + b + c = 0$. But then $b^2 + ba + bc = b(a + b + c) = 0$, so $b$ is the other root of $x^2 + ax + bc = 0$, i.e., $\beta = b$. Similarly, $a^2 + ab + ac = a(a + b + c) = 0$, so $a$ is the other root of $x^2 + bx + ca = 0$, i.e., $\alpha = a$. Then $\alpha = a$ and $\beta = b$ are roots of the equation $x^2 - (a + b)x + ab = 0$, and since $c = -(a + b)$, they are roots of $x^2 + cx + ab = 0$.

**5.** Find all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $x^2 f(x) + f(1 - x) = 2x - x^4$ for all $x \in \mathbb{R}$.

**Solution:** Since $x^2 f(x) + f(1 - x) = 2x - x^4$ for all $x \in \mathbb{R}$, we also get

$$(1 - x)^2 f(1 - x) + f(x) = 2(1 - x) - (1 - x)^4$$

for all $x \in \mathbb{R}$.

Eliminate $f(1 - x)$ from the above equalities and get $f(x) = 1 - x^2$.