

Solutions to round 1

1. Let E and L denote the number of pieces of candy that Eric and Lisa will have after the game is over. Then, obviously,

$$0 < E < 101, \quad 0 < L < 101 \quad \text{and} \quad E + L = 101.$$

If E and L have a common divisor $0 < d$, then d will also divide 101. But 101 is a prime number and so $d = 1$ or $d = 101$. Hence $d = 1$ and so E and L are relatively prime. Therefore, regardless of the strategy, Lisa will always win the game.

2. Let us count the years that are not lucky. Let $1 \leq N \leq 99$. Assume that $N \neq MD$ where $1 \leq M \leq 12$, and $1 \leq D \leq 31$. Then N has to have a prime factor > 31 . There are 14 primes between 32 and 99 namely,

$$37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$$

Moreover, the 4 numbers 2×37 , 2×41 , 2×43 , and 2×47 cannot be written as MD . January and March have 31 days and so the years 2031 and 2093 are lucky years ($93 = 31 \times 3$), but the year 2062 is not a lucky year ($62 = 2 \times 31$) as February has only 28 (or 29) days. The year 2058 ($2 \times 29 = 58$) is not a lucky year because 2058 not divisible by four, and so February in that year has only 28 days. 2100 cannot be a lucky year and so we have a total of

$$100 - 14 - 4 - 1 - 1 - 1 = 79$$

lucky days.

3. Let P be the intersection of AA' , BB' , and CC' . Obviously,

$$\text{Area}AC'P = \text{Area}C'BP = a,$$

$$\text{Area}PBA' = \text{Area}PA'C = b,$$

$$\text{Area}APB' = \text{Area}B'PC = c.$$

Moreover,

$$a + 2c = a + 2b, \quad \text{and} \quad b + 2c = b + 2a$$

and so $a = b = c$.

4. Let

$$N = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 100a_2 + 10a_1 + a_0,$$

where $0 \leq a_i \leq 9$ (here we can replace 9 by 1 but this is not necessary) are the digits in the decimal representation of N . We have

$$a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 = 3.$$

We have

$$\begin{aligned} N &= 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 100a_2 + 10a_1 + a_0 - (a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0) + 3 \\ &= 99 \cdots 9a_n + \cdots + 99a_2 + 9a_1 - 3. \end{aligned}$$

we deduce that N is divisible by 3 but not by 9. A perfect square that is divisible by 3 is also divisible by 9. Therefore, n cannot be a perfect square.