NDSU Mathematics Talent Search 2001-2002
Solutions

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1. **It takes 8 hours less to fill a pond using a large pipe than it takes using a smaller pipe. If, when used together, they fill the pond in 3 hours, how long would it take each to fill it alone?**

   Let t be the number of hours it takes the small pipe to fill the pond, so it takes the large pipe $t-8$ hours to fill the pond. Therefore, the amount of water (measured as a fraction of the entire pond) that flows through the smaller pipe in an hour is $\frac{1}{t}$. The amount of water that flows through the larger pipe in an hour is $\frac{1}{t-8}$. When both pipes are working, the total amount of water that goes into the pond each hour is

   $$\frac{1}{t}+\frac{1}{t-8}=\frac{2t-8}{t(t-8)}=\frac{1}{3},$$

   the last equality holding since it takes 3 hour to fill the pond. Cross multiplying gives

   $$3(2t-8)=t(t-8)$$

   which simplifies to

   $$t^2-14t+24=0.$$

   Then the quadratic formula yields

   $$t=\frac{14\pm\sqrt{196-96}}{2}=7\pm 10=17 \text{ or } -3,$$

   but since a negative answer makes no sense in this setting, $t=17$. Hence the smaller pipe takes 17 hours to fill the pond, while the larger pipe takes only 9 hours.

2. **Let $A_1, A_2, A_3, \ldots, A_n$, represent an arbitrary arrangement of the numbers 1, 2, 3, ..., n. Prove that if n is odd, the product**

   $$(A_1-1)(A_2-2)(A_3-3)\ldots(A_n-n)$$

   is an even number.

   The jth factor in the product, (A_j-j), is even if either both A_j and j are even or both are odd (i.e. if they have the same parity). The factor will be odd if A_j and j have different parity.

   Since n is odd, there are more odd numbers between 1 and n than there are even numbers, so it is impossible to pair off each odd j with an even A_j. Hence there must be some pair of j and A_j that have the same parity. This means one of the factors (A_j-j) is even. Since one factor is even then the whole product must also be even.

3. **How many positive integers of n digits exist such that each digit is 1, 2, or 3? How many of these contain all three of the digits 1, 2, and 3 at least once?**

Let N be the set of positive integers with n digits, each of which is 1, 2, or 3. Suppose we are randomly choosing an element of N. Since, for each digit, we can choose out of three possibilities, and since each digit choice is independent of the others, N must have we have a total of 3^n numbers.

Now consider the set A of positive integers with n digits, each of which is 1 or 2. By the similar reasoning as above, this set contains 2^n numbers. Let B and C be the sets of positive integers with n digits, each of which is 1 or 3, or 2 or 3, respectively. Again, each of these sets contains 2^n numbers.

The sets A and B have only one number in common, namely $111\ldots111$. Similarly, A and C only have the number $222\ldots222$ in common and B and C only have the number $333\ldots333$ in common. Therefore the union of the three sets have a total of $2^n+2^n+2^n-3=3(2^n-1)$ elements; the negative three comes from the fact that exactly three numbers are counted twice.

Those numbers in N which contain all three of the digits 1, 2, and 3 at least once are exactly those that do not form part of A or B or C. Thus, there are exactly $3^n-3(2^n-1)=3(3^{n-1}-2^n+1)$ of these numbers.

4. **The lengths of sides CB and CA of triangle ABC are a and b, respectively, and the angle between them measures $120°$. Express the length of the bisector to angle C in terms of a and b.**

Place C at the origin of the cartesian plane and rotate so that the angle bisector points in the direction of the y-axis, say with B $60°$ to the right and A $60°$ to the left. The distance from C to B is a and the segment between them makes a $30°$ angle with the positive x-axis, so the coordinates of point B must be $(a\cos 30°, a\sin 30°)=(\frac{\sqrt{3}a}{2},\frac{a}{2})$. Similarly, side CA is length b and makes a $150°$ angle with the positive x-axis, so the coordinates of the point A is $(b\cos 150°, b\sin 150°)=(-\frac{\sqrt{3}b}{2},\frac{b}{2})$.

Consider the line between these A and B. Since we placed the angle bisector so it points in the direction of the y-axis, the length of the bisector is exactly the y-intercept of this line. A simple computation then shows that this line has slope

$$m=\frac{\frac{a}{2}-\frac{b}{2}}{\frac{\sqrt{3}a}{2}+\frac{\sqrt{3}b}{2}}=\frac{a-b}{\sqrt{3}(a+b)},$$

so its y-intercept is

$$\frac{a}{2}-m\frac{\sqrt{3}a}{2}=\frac{a}{2}-\frac{a-b}{\sqrt{3}(a+b)}\frac{\sqrt{3}a}{2}=\frac{a}{2}-\frac{a(a-b)}{2(a+b)}=\frac{ab}{a+b}.$$

5. **A single square is removed from a 2^N by 2^N checkerboard. Show that (regardless of which square is missing) the checkerboard can be tiled with non-overlapping pieces of the shape shown in the figure. (Each square is the size of one of the squares of the checkerboard and the pieces can be rotated).**

   If $N = 1$ then we are looking at a 2 by 2 checkerboard. removing any one of the squares gives a board which looks exactly like one of our tiles. Therefore we can tile this checkerboard using a single tile.

   Suppose that $N > 1$ and consider the four 2^{N-1} by 2^{N-1} checkerboards you get by cutting in half horizontally and vertically. Call these quarter boards A_1, A_2, A_3, and A_4. Exactly one of these is missing a square; suppose that's A_1. Place a single tile around the common corner of the four quarter boards so it covers exactly one square from A_2, A_3, and A_4. With this tile in place, each of the quarter boards will be missing a single square and can be tiled by repeating this process until get down to 2 by 2 checkerboards with one piece missing.

6. **Start with a randomly chosen positive integer and take its square root. Double the result and take the square root again. Continue doubling and taking square roots indefinitely. What value are you getting closer and closer to? Explain why.**

   Let a_0 be the first number you choose. After doubling and taking the square root once, we get

$$a_1 = \sqrt{2a_0} = 2^{\frac{1}{2}} {a_0}^{\frac{1}{2}}.$$

   Doubling and taking the square root a second time yields

$$a_2 = \sqrt{2\sqrt{2a_0}} = 2^{\frac{3}{4}} {a_0}^{\frac{1}{4}}.$$

   Each time we double and take the square root, we end up halving the power of a_1 and adding one half to half the power of 2, so after n iterations we have

$$a_n = \sqrt{2 \ldots \sqrt{2\sqrt{2a_0}}} = 2^{\frac{2^n-1}{2^n}} {a_0}^{\frac{1}{2^n}}.$$

   As n increases, $\frac{1}{2^n}$ gets closer and closer to 0 while $\frac{2^n-1}{2^n}$ gets closer and closer to 1. Thus in the limit our numbers get closer and closer to

$$\lim_{n\to\infty} a_n = 2^1 {a_0}^0 = 2.$$

7. **I was about to get \$0.83 in change at the Math Emporium last night when there was a blackout and the entire store became pitch black. At the moment when the lights went out, the cashier was handing me eight coins (quarters, dimes, nickels, and pennies). How do I know he gave me the wrong amount of change?**

The least number of coins (quarters, dimes, nickels, and pennies) that can be used to give \$0.83 in change is seven: 3 quarters, 1 nickel, and 3 pennies. This is the combination of coins that a well-trained cashier will give you. However, we can change the number of coins of each denomination by (i) exchanging one quarter for two dimes and a nickel, (ii) exchanging one dime for two nickels, or (iii) exchanging one nickel for five pennies. Any other change in the number of coins can be derived as a combination of these three "moves". Note that move (i) increases the number of coins by 2, while move (ii) increases the number of coins by 1 and move (iii) increases the number of coins by 4. Since the ideal situation consists of 7 coins, the only way to increase to 8 is by using move (ii); however this move is not available to us because we have no dimes. Thus it is impossible to give \$0.83 in 8 coins.

8. **For what values of c does the formula**

$$x^3 + cx^2 + cx + 1 = 0$$

have exactly one solution?

Notice that

$$(-1)^3 + c(-1)^2 + c(-1) + 1 = -1 + c - c + 1 = 0.$$

This means that -1 is always a solution. This also means that the polynomial $x^3 + cx^2 + cx + 1$ is divisible by $x + 1$. Doing the long division gives

$$x^3 + cx^2 + cx + 1 = (x + 1)(x^2 + (c - 1)x + 1).$$

Since a product is zero exactly when one of its factors is, the cubic equation above will have exactly one solution when the quadratic equation

$$x^2 + (c - 1)x + 1 = 0$$

has none. By the quadratic formula, this will happen when

$$B^2 - 4AC = (c - 1)^2 - 4 < 0.$$

By solving for x, this inequality reduces to

$$-1 < c < 3.$$