NDSU Mathematics Talent Search 2000-2001
Solutions

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1. **How many triples of positive integers (x, y, z) are there for which $28x + 30y + 31z = 365$?**

   Recall that 365 is the number of days in a non-leap year. In such a year there are exactly 7 months with 31 days (January, March, May, July, August, October, and December), 4 months with 30 days (April, June, September, November), and 1 month with 28 days (February). Therefore $(1, 4, 7)$ is one possible solution to the equation.

   To get other solutions, notice that we can turn a 30-day month into a 28-day month, giving us two leftover days in that year. Of course, we can use these leftover days to turn two 30-day months into 31-day months. This means that if (x, y, z) is a solution, then so is $(x + 1, y - 3, z + 2)$. Thus $(2, 1, 9)$ is another solution. Unfortunately, we can't repeat this process or we will run out of 30-day months.

   We can also try to do the opposite, and turn a 28-day month into a 30-day month by borrowing a single day from each of two 31-day months. Thus, if (x, y, z) is a solution, then so is $(x - 1, y + 3, z - 2)$. If we start with $(2, 1, 9)$ this process will take us to $(1, 4, 7)$, and of course we can't repeat or we will run out of 28-day months.

   Therefore there are only two triples of positive integers which satisfy the equation.

2. **Over the top of a fence is placed a rope, with the same amount hanging on each side. The rope weighs $\frac{1}{3}$ pound per foot. On one end of the rope hangs a monkey holding a banana; on the other end is a weight weighing the same amount as the weight of the monkey.**
   The banana weighs 2 ounces per inch. The length of the rope (in feet) is equal to the age of the monkey, and the weight of the monkey (in ounces) is as much as the age of the monkey's mother. The combined ages of the monkey and it's mother is thirty years.
   The weight of the banana plus one half the weight of the monkey is one quarter as much as the sum of the weights of the weight and the rope, where all weights are in the same units.
   The monkey's mother is one half as old as the monkey will be when it is three times as old as it's mother was when she was one half as old as the monkey will be when it is twice as old as it is now.
   How long is the banana?

   We are looking for b, the length of the banana (in inches). Let r be the length of the rope (in feet). Since the rope weighs $\frac{1}{3}$ pound per foot, the weight of the rope (in ounces) is $\frac{16}{3}r$. Similarly, the banana weighs 2 ounces per inch, so the weight of the banana (in ounces) is $2b$.

Now, the monkey's age is equal to the length of the rope (in feet), in other words, r. Let m be the age of the monkey's mother, so the weight of the monkey (in ounces) is also m. The combined ages of the monkey and it's mother is thirty years, so

$$r + m = 30. \tag{1}$$

Next, we are told that the weight of the banana ($2b$) plus one half the weight of the monkey ($\frac{1}{2}m$) is one quarter as much as the sum of the weights of the weight (which weights the same as the monkey, so m) and the rope ($\frac{16}{3}r$). This means that

$$2b + \tfrac{1}{2}m = \tfrac{1}{4}(m + \tfrac{16}{3}r). \tag{2}$$

Finally, the monkey's mother is one half as old as the monkey will be when it is three times as old as it's mother was when she was one half as old as the monkey will be when it is twice as old as it is now. The monkey will be $2r$ years old when it is twice as old as it is now. Half that age is r. Three times as old as that is $3r$. Half as old as that is $\frac{3}{2}r$. Since this is how old the monkey's mother is, $m = \frac{3}{2}r$. Putting this into equation (1) gives

$$r + \tfrac{3}{2}r = \tfrac{5}{2}r = 30,$$

so $r = 12$ and $m = 18$. Then, by equation (2),

$$b = \tfrac{1}{2}(\tfrac{1}{4}(m + \tfrac{16}{3}r) - \tfrac{1}{2}m) = \tfrac{1}{2}(\tfrac{1}{4}(18 + \tfrac{16}{3}12) - \tfrac{1}{2}18) = \tfrac{23}{4}.$$

This means that the banana is 5 and three quarter inches long.

3. **Suppose that P and Q are two distinct positive integers, and consider the set of whole number combinations of P and Q. For example 0, $2P$, and $3P + 5Q$ are all examples of whole number combinations of P and Q. Let $S(P,Q)$ denote the set of all possible whole number combinations of P and Q. Then, for example, $S(2,3) = 0,2,3,4,...$ is the set of all whole numbers not equal to 1, while $S(2,6) = 0,2,4,6,...$ is the set of all even whole numbers.**
Determine, as a function of N, how many whole numbers are NOT in $S(N, N+1)$.

Let $S = S(N, N+1)$. Notice that 0 is always an element of S but that the next $N-1$ integers (from 1 through $N-1$) are not. Similarly, N and $N+1$ will both be in S. Since the next whole number combination of N and $N+1$ doesn't come until $2N$, this leaves a gap of $N-2$ integers (from $N+2$ through $2N-1$) before we get to $2N, 2N+1 = 1(N)+1(N+1)$, and $2N+2 = 2(N+1)$, all three of which form part of the set S. This block of 3 consecutive elements of S is followed by a gap of $N-3$ whole numbers not in S ($2N+3$ to $3N-1$), which in turn is followed by followed by four integers from S: $3N, 3N+1 = 2(N)+1(N+1), 3N+2 = 1(N)+2(N+1)$, and $3N+3 = 3(N+1)$. The pattern continues in this way with the blocks of integers from S getting larger and larger and the gaps in between these getting smaller and smaller. Eventually, the size of the gaps reaches zero and the blocks begin to overlap. By this point, we will have seen

$$(N-1)+(N-2)+(N-3)+\ldots+1 = \frac{N(N-1)}{2}$$

whole numbers not in S.

4. **We place five identical spheres in a rectangular box as shown in the figure. The bottom of the box is a 10 cm by 10 cm square. The spheres are packed as tight as possible, with the center sphere touching the other four, and all five spheres touching the top and bottom of the box. How tall is the box? (Give both an exact answer and a decimal approximation).**

Consider the horizontal rectangle which cuts the box in two equal halves. This rectangle contains the centers of the five spheres. In fact, if we place our x and y axes appropriately, with the origin at one of the corners of this rectangle, the coordinates of the centers of the spheres are $(r,r), (10-r,r), (5,5), (r,10-r)$, and $(10-r,10-r)$, assuming of course that r is the radius of all five spheres. Now the distance from a corner of this rectangle to the center of its nearest sphere is $\sqrt{r^2+r^2} = \sqrt{2}r$. On the other hand, the distance from the center of one of the outside spheres to the center of sphere on the opposite side of the box is $4r$. Since the length of the diagonal of the rectangle equals the distance from a corner of the rectangle to the center of the nearest sphere, together with the distance from that center to the center opposite it, followed by the distance from that center to its nearest corner, we have

$$10\sqrt{2} = \sqrt{2}r + 4r + \sqrt{2}r = (4+2\sqrt{2})r.$$

Hence the radius of the spheres is

$$r = \frac{10\sqrt{2}}{4+2\sqrt{2}} = \frac{10\sqrt{2}(4-2\sqrt{2})}{16-8} = \frac{40\sqrt{2}-40}{8} = \frac{10\sqrt{2}-10}{2}.$$

Then the height of the box is twice this radius:

$$h = 10\sqrt{2} - 10 \approx 4.1421\text{cm}.$$

5. **Find two positive integers M and N so that:**
   (i) $M/N = 1/2$, **and**
   (ii) **the union of the digits in the decimal (base 10) representation of** M **and** N **consists of** $\{1,2,\ldots,9\}$ **with no repeats.**

There are 10 possible solutions, as indicated by the following table:

M	N	M	N
6729	13458	7692	15384
6792	13584	7923	15846
6927	13854	7932	15864
7293	14586	9273	18546
7329	14658	9327	18654

6. **In a normal game of Tic-Tac-Toe, player X wins if she places three X's in the same row, in the same column, or in a diagonal. In a game of "wrap-around" Tic-Tac-Toe, a player can also wrap around the edges of the 3x3 grid to complete a winning three-in-a-row. Therefore, we can think of the top three squares in the grid as being directly below the bottom three squares and the three leftmost squares as being directly to the right of the three rightmost squares. If in a normal game of Tic-Tac-Toe, there are eight ways for player X to win, how many ways are there in wrap-around Tic-Tac-Toe?**

In a regular game of Tic-Tac-Toe, the eight ways to win consist of 3 horizontals, 3 verticals, and 2 diagonals. In wrap-around Tic-Tac-Toe, every position is part of a unique horizontal, a unique veritcal, and 2 different diagonals. Since there are 9 squares in the grid and each one offers 4 ways to win, but each way of winning is counted three times (once for each of the X's in the three-in-a-row), there is a total of

$$4 \times 9/3 = 12$$

ways to win in wrap-around Tic-Tac-Toe.

7. **An NDSU mathematics major paddled six miles upstream on the Red River of the North, at which point his hat fell into the river. Without stopping, he continued to paddle upstream at the same rate for two more hours. Then he turned and paddled back to the starting point, arriving at exactly the same time as his hat, which had floated downstream after falling off. How fast was the river flowing?**

Suppose that the river is flowing at x mph, that the math major paddles consistently at y mph, and that it takes him t hours to row back from the point he turns around. We can think of the math major's trip as three different distances.

First of all, he travels 6 miles upstream, at which point he drops his hat. The hat travels back downstream at the speed of the river for $t + 2$ hours before he catches up with it back at the starting line. This means that the total distance covered in this first leg of the journey is $x(t + 2) = 6$.

Next comes the distance he travels upstream after dropping his hat. Since he paddles against the currents at a speed of $y - x$ mph, the total distance covered in these 2 hours is $2(y - x)$.

Finally, on the way downstream, he travels the entire distance in t hours at a speed of $y + x$ mph. This means that the distance he travels downstream must be $t(y + x)$. Setting this length equal to the sum of the previous two gives us the equation

$$t(y + x) = x(t + 2) + 2(y - x).$$

By multiplying out and cancelling, we can simplify this equation down to

$$ty = 2y.$$

Of course, y cannot be zero (otherwise the math major would have gotten nowhere!) so we can divide by y and obtain $t = 2$. Thus it took the hat 4 hours to flow downstream 6 miles, and hence the river is flowing at $\frac{6}{4} = 1.5$ mph.

8. **Suppose that I have a whole bunch of poker chips, each one with a different positive integer written on it. Before eating my turkey dinner on Thanksgiving Day, at one minute before five, I put nine chips, labelled 1 through 9, in a large sack. At a half a minute before five, I add ninety chips, labelled 10 through 99, to the sack and immediately take out chip number 1. At a quarter of a minute before five, I place nine hundred chips labelled 100 through 999 in the sack and remove chip number 2. Then one eighth of a minute before five I add nine thousand chips labelled 1000 to 9999 and remove chip number 3. Assuming that I can continue to do this (so I can move arbitrarily fast) and I don't run out of chips or room in the sack, how many chips will there be in the sack when I sit down to eat dinner at five?**

   The sack will be empty by the time I sit down to dinner. This is because every chip that gets put into the sack, gets taken out at some point. In particular, the chip labelled n will be taken out of the sack 2^{-n} minutes before 5.

9. **In a game of American Football, most of the points scored come from field goals and touchdowns. A field goal is worth 3 points. A touchdown is worth 6 points plus the opportunity to score an extra point; thus most touchdowns are worth 7 points. What is the largest number that cannot be a valid score in a football game in which all points are scored from touchdowns and field goals (i.e. no safeties or two-point conversions)?**

Consider the remainders obtained when dividing a possible score by 7. Clearly all those scores which are multiples of 7 will be legitimate scores, as they represent a whole number of 7-point touchdowns. Similarly, all those scores with remainder 3 or 6 will also be valid, as they represent a whole number of touchdowns plus one or two field goals.

Among those scores with remainder 1, 15 is the first multiple of 3. Thus every score with remainder 1 which is 15 or higher is a valid score; for instance, $22 = 5*3+7$ = five field goals plus one (7-point) touchdown. Among those scores with remainder 2, 9 corresponds to 3 field goals, so every score with remainder 2 which is 9 or higher is also a valid score. Similarly, the first multiple of 3 which has remainder 4 when divided by 7 is 18; therefore, every score with remainder 4 which is 18 or higher is a valid score. Finally, 12 is the first score with remainder 5 which is a multiple of 3, so any score with remainder 5 which is at least as large as 12 will correspond to a valid score.

This leaves 1, 2, 4, 5, 8. and 11 as the only impossible scores in a football game. Of these, 11 is the largest.

10. **Nine coins look alike but one is a counterfeit and weighs less than the others. Describe how, using only a balance, you can find the fake in only two weighings.**

Separate the nine coins into three groupings of three coins each. Call these A, B, and C. Weigh A against B on the balance. If either of the two groupings is lighter than the other, then it contains the counterfeit; if they weigh the same, the counterfeit is in group C.

In this way, with one weighing, we know that the counterfeit is among three coins, say a, b, or c. Weigh a against b on the balance. If either of the two coins is lighter, it is the counterfeit; if they weigh the same, then the c is the counterfeit.

11. **A square is inscribed in a circle which is circumscribed by an equilateral triangle, as indicated by the figure below. How many times larger is the area of the triangle as that of the square?**

Suppose that the radius of the circle is r. Then the center of the circle and any two vertices of the circumscribed equilateral triangle form an isosceles 30-120-30 triangle with altitude r. The base of this triangle must then measure $2\frac{r}{\tan 30^o} = 2\sqrt{3}r$. However, this base is just one of the sides of the circumscribed equilateral triangle. The Pythagorean Theorem implies that the altitude of an equilateral triangle measures $\frac{\sqrt{3}}{2}$ times its base. Thus, the area of the triangle is $\frac{1}{2}(2\sqrt{3}r)(\frac{\sqrt{3}}{2}2\sqrt{3}r) = 3\sqrt{3}r^2$.

On the other hand, the center of the circle and any two adjacent vertices of the inscribed square form an iscosceles right triangle whose legs measure r. The hypotenuse of this triangle must then measure $\sqrt{2}r$. However, this is just one of the sides of the inscribed square, whose area is then $(\sqrt{2}r)^2 = 2r^2$.

The ratio of the area of the triangle to that of the square is then

$$\frac{3\sqrt{3}r^2}{2r^2} = \frac{3\sqrt{3}}{2}.$$

12. **A geography class at NDSU contains 8 men and 7 women. If the professor selects 3 people at random to work on a presentation on the economic effects in Bolivia arising from its being a landlocked country, what is the probability that the number of men selected exceeds the number of women selected.**

There are 15 students in total, so there are

$$\frac{15 \cdot 14 \cdot 13}{3!} = 455$$

possible choices for the 3-person committee. There are two distinct ways in which the number of men selected is greater than the number of women selected: either only men are selected or two men and one woman are selected. Since there are 8 men in the class,

$$\frac{8 \cdot 7 \cdot 6}{3!} = 56$$

of the 455 possible committees consist entirely of men. In addition,

$$\frac{8 \cdot 7}{2!} \cdot 7 = 196$$

possible committees consist of 2 men and a single woman. Hence the probability that more men are selcted than women is

$$P = \frac{56 + 196}{455} = \frac{252}{455} = \frac{36}{65} \approx .55385.$$

13. **What is the smallest whole number N whose decimal expansion ends in a "3" and has the property that when we move that "3" to the front of the decimal expansion and shift all of the other digits down by one position we get 3N?**

When we multiply N by 3, we multily each of its digits by 3, and then add "carries" of 0, 1 or 2 to the digit on the left. Since the last (rightmost) digit in N is 3, the last digit in $3N$ must be a 9. This means that the last two digits in N are 93, making the last two digits of $3N$ equal to the last two digits of $3 \times 93 = 279$. In other words, the decimal expansion for $3N$ ends in 79. This makes the last three digits in N 793, so , since $3 \times 793 = 2379$, the last three digits in $3N$ are 379. Now the last four digits of N are 3793. We continue in this fashion, obtaining one more digit each time. We will have reached the smallest possible N with this property when we obtain a digit of 1 which gets a carry of zero when multiplied by 3. This happens after 28 iterations, when we reach the number

$$N = 1,034,482,758,620,689,655,172,413,793.$$

Multiplication by 3 yields

$$3N = 3,103,448,275,862,068,965,517,241,379,$$

as desired.

14. **Next year, 2002, is a palindrome year because it reads the same forwards and back. The next palindrome year after that will be 2112. 2002 and 2112 are a whole 110 years apart. After the year 1000, what is the shortest period of time separating two palindrome years?**

If we only consider four-digit years, then clearly the first two digits of the year determine the other two. This means we see one palindrome year in every century. However, the position of that year within each century shifts to later and later in the century as time goes by. For instance, the first few palindrome years after the year 1000 were: 1001, 1111, 1221, and so on. With in each century, each occurs ten years after the previous, and hence the separation between these is of 110 years. This is the case until 1991, after which the palindrome moves to the begining of the century again. Hence, 1991 is followed by 2002, only eleven years later. The next time the palindrome moves back to the start of its century will be in 3003, again eleven years after 2992. Therefore the shortest amount of time between two palindrome years between the years 1000 and 9999 is 11.

The next palindrome year after 9999 occurs only two years later, in 10001. This is the shortest period of time possible between two palindrome years, bar none. Otherwise we would have two consecutive years, both of which were palindromes. Since the last digits of any two consecutive integers are different, the first digits must also be different. But this means that the greater of the two years is a multiple of 10, making its last digit a zero. But these years are never considered palindromes, since we rarely write leading zeroes.

15. **Suppose we made a rope lasso long enough to encircle the entire world at the equator. How much longer would it have to be so that the lasso could be loosened up and lifted up one kilometer off the face of the earth?**

Suppose that the radius of the earth is r kilometers. Then the circumference at the equator is $2\pi r$ kilometers. If we lifted the rope by 1 km, then the radius of the lasso would be $r+1$ and its total length would be $2\pi(r+1) = 2\pi r + 2\pi$. This means the lasso would only have to be $2\pi \approx 6.28$ km, or about 3.9 miles, longer than the equator.

16. **The Prince of Bombay has a beautiful garden in the middle of which sits the largest mango tree imaginable. This tree gives the biggest, best tasting mangoes in the world, each one weighing about two pounds! To keep the tree and its fruit all to himself, the Prince builds seven concentric walls surrounding the garden. Each wall has a single gate with a eunuch guarding the gate. In order to get to the mango tree, you would have to go past all seven guards.**

    One day a small man approaches the outside guard and proposes the following deal: If the guard lets him past the gate and he can get to the mango tree then the man promises to give the guard one less than half of the mangoes he is carrying on his way out. The outside guard accepts the deal and lets him through. The man then goes up to the second guard, makes the same deal with him and gets past the second gate. He does the same bit of convincing five more times and eventually gets to the mango tree.

    If no mangoes are to be split in half, how many mangoes must the small man take from the tree so that he can make good on his seven promises?

Suppose that, after paying off the outside guard, the man walks away with 3 mangoes. This means that 3 is one more than half of the mangoes he had when he approached that guard. In other words, after he paid off the sixth guard, the man had $2(3-1) = 4$ mangoes. He, therefore, must have been carrying $2(4-1) = 6$ mangoes when he walked away from fifth guard, $2(6-1) = 10$ mangoes when he walked away from fourth guard, $2(10-1) = 18$ mangoes when he walked away from third guard, $2(18-1) = 34$ mangoes when he walked away from second guard, $2(34-1) = 66$ mangoes when he walked away from first guard, and a whopping $2(66-1) = 130$ mangoes when he walked away from the tree. Now remember that each mango weighs about 2 pounds. This means the small man must have been carrying 260 pounds of mango – way too much for a small man to carry on his own.

Suppose, then, that after paying off the outside guard, the man walks away with 2 mangoes. This means that 2 is one more than half of the mangoes he had when he approached that guard. In other words, when he approached the last guard, the man had $2(2-1) = 2$ mangoes. By the same logic as above, the man picked up two mangoes art the tree and then, true to his deal with each guard, gave each of them one less than he was carrying: $\frac{1}{2}2 - 1 = 0$. Since he kept his part of the bargain, the guards had to let him through for free. I'm sure they never fell for that trick ever again!