Numerical Monoids are Totally Cool! [1]

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Abstract

Several years ago, McDonald's Chicken McNuggets could be purchased only in packs of $6, 9$ and 20. Under this restriction, it is not possible to order exactly 43 McNuggets, but there are 2 different ways to order 44 McNuggets: $4\cdot 6+0\cdot 9+1\cdot 20 = 44$ and $1 \cdot 6 + 2 \cdot 9 + 1 \cdot 20 = 44$. This computation is a modern version of an older problem commonly known as the *postage stamp problem*.

The ideas in the first paragraph can be mathematically formalized as follows. Let $n_1, n_2, \ldots, n_t$ be a finite set of positive integers. Then the numerical monoid S generated by $\langle n_1, n_2, \ldots, n_t\rangle$ is the set of all s such that $s = \sum a_i n_i$, where each a_i is a nonnegative integer. If the integers $n_1, n_2, \ldots, n_t$ are relatively prime, then it can be shown that there is a largest integer m such that $m \notin S$. This m is known as the *Frobenius number* of S and because of the difficulty in its computation, has a long and elusive mathematical history. In addition to discussing the Frobenius number, we shall also talk about how elements factor in a numerical monoid. For $s = \sum a_i n_i$ in S, $\sum a_i$ is called a *factorization length* of s. Let $\mathcal{L}(s) = \{m_1, m_2, \ldots, m_k\}$ (where $m_i < m_{i+1}$ for $1 \leq i \leq k-1$) be the set of all possible factorization lengths of s. The sets $\mathcal{L}(s)$ have interesting properties, and I will discuss in some details results concerning these sets obtained by my past REU students.

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[1] The title for this talk is a quote from one of my former REU students Nathan Kaplan who is currently at the Tripos III at Cambridge.