

A CHARACTERIZATION OF POLYNOMIAL RINGS WITH THE HALF-FACTORIAL PROPERTY

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1. INTRODUCTION

In this paper, R will be an integral domain with quotient field K and x an indeterminate. Following Zaks [4], we define R to be a half-factorial domain (HFD) if R is atomic, and given any two irreducible factorizations of an element $a \in R$

$$a = \pi_1 \pi_2 \dots \pi_k = \xi_1 \xi_2 \dots \xi_m,$$

then $k = m$.

Unlike unique factorization domains (UFD's), HFD's do not behave well with respect to polynomial extensions in general (see [1], [2, Example 5.4]). In [4, Theorem 2.4] it was established that if R is a Krull domain with divisor class group $\text{Cl}(R)$, then $R[x]$ is an HFD if and only if $|\text{Cl}(R)| \leq 2$. However, there exist Krull domains with the HFD property that have $|\text{Cl}(R)| > 2$, so in general, the HFD property is lost in the polynomial extension.

We shall presently show that for $R[x]$ to be an HFD, it is necessary for R to be integrally closed, and from this we will deduce

a characterization for “polynomially stable” Noetherian HFD’s.

2. THE MAIN THEOREM

Let R and K be as above. We first record a useful lemma.

LEMMA 2.1. Let $p(x)$ be irreducible in $R[x]$, and let $0 \neq r \in R$. If $rp(x) = r_1 r_2 \dots r_t f_1 f_2 \dots f_k$ with $r_i \in R$ for $1 \leq i \leq t$ and $f_i \in R[x]$ with $0 < \deg(f_i) < \deg(p)$ for $1 \leq i \leq k$, then no f_i is monic.

Proof: Suppose that $rp(x) = r(q_{n+m}x^{n+m} + q_{n+m-1}x^{n+m-1} + \dots + q_1x + q_0) = (r_nx^n + \dots + r_0)(x^m + s_{m-1}x^{m-1} + \dots + s_0) = g_1(x)g_2(x)$ with $n \geq 1$.

From this we obtain the following system of equations:

$$\begin{aligned} r_n &= r q_{n+m} \\ r_{n-1} + r_n s_{m-1} &= r q_{n+m-1} \\ &\vdots \\ &\vdots \\ &\vdots \\ r_0 + r_1 s_{m-1} + \dots + r_m s_0 &= r q_m \end{aligned}$$

Inductively from these equations, we get that $r|r_i$ for every i . Therefore, $g_1(x) = rg(x)$ with $g(x) \in R[x]$. This shows that $p(x) = g(x)g_2(x)$, which is a contradiction.

With this lemma in hand, we can now prove the main theorem of this paper.

THEOREM 2.2. Let R be an integral domain. If $R[x]$ is an HFD, then R is integrally closed.

Proof: Assume that R is not integrally closed. We shall show that $R[x]$ is not an HFD. We note that we can also assume that R is an HFD, for if not, then $R[x]$ is certainly not an HFD (see [1]).

Let K be the quotient field of R , and let $\omega \in K \setminus R$ such that ω satisfies the monic irreducible polynomial $p(x) = x^n + p_{n-1}x^{n-1} + \dots + p_1x + p_0 \in R[x]$. Also assume that $\omega = r/s$ with $r, s \in R$ such that r and s have no factor in common (which is valid since R is an HFD). Consider the following element of $R[x]$:

$$\begin{aligned} s^n p(x) &= s^n x^n + p_{n-1} s^n x^{n-1} + \dots + p_1 s^n x + p_0 s^n \\ &= (sx - r)q(x) \text{ with } q(x) \in R[x]. \end{aligned}$$

By assumption we have the following facts.

1. The number of factors of one irreducible factorization of the left hand side is $mn + 1$, where m is the number of irreducible factors of s .
2. The polynomial $(sx - r)$ is irreducible.

So we will investigate the number of factors of $q(x)$.

Notice that the leading coefficient of $q(x)$ is s^{n-1} . Assume that $q(x) = f_1(x) \dots f_k(x) r_1 \dots r_t$ where each $f_i \in R[x]$ is irreducible of positive degree and each r_i is irreducible in R . As $p(x)$ is irreducible in $R[x]$, Lemma 2.1 shows that none of the f_i 's is monic, and so we obtain the equation

$$s^{n-1} = L_1 \dots L_k r_1 \dots r_t,$$

where L_i is the leading coefficient of $f_i(x)$ and is a nonunit.

As R is an HFD, we have that $k + t \leq m(n - 1)$. We conclude that the number of irreducible factors of $s^n p(x)$ (from this point of view) is $k + t + 1 \leq m(n - 1) + 1 \leq mn + 1$. For $R[x]$ to be an HFD, the last inequality must be an equality, and hence $m=0$. This contradicts the fact that $\omega \in K \setminus R$.

We now give a corollary to this theorem which completely classifies all Noetherian HFD's that have "polynomial stability".

COROLLARY 2.3. Let R be a Noetherian ring. Then the following conditions are equivalent:

1. R is a Krull domain with $|\text{Cl}(R)| \leq 2$.
2. $R[x]$ is an HFD.
3. $R[x_1, \dots, x_n]$ is an HFD for all $n \geq 1$.
4. $R[x_1, \dots, x_n]$ is an HFD for some $n \geq 1$.

Proof: We first observe that the implications (3) implies (4) and (4) implies (2) are obvious. We will show that (1) implies (3) and (2) implies (1).

For the implication (1) implies (3), since R is a Krull domain with $|\text{Cl}(R)| \leq 2$, then $R[x_1, \dots, x_n]$ is also a Krull domain with $|\text{Cl}(R[x_1, \dots, x_n])| = |\text{Cl}(R)|$ (cf [3, Chapter 7, Proposition 13]). Since if R is a Krull domain, then $R[x]$ is an HFD if and only if $|\text{Cl}(R)| \leq 2$ [4, Theorem 2.4], we obtain the result inductively.

For (2) implies (1), we assume that $R[x]$ is an HFD. The previous theorem shows that R is integrally closed and hence a Krull domain (as R is Noetherian). Once again applying a result of Zaks [4, Theorem 2.4], we obtain that R must have $|\text{Cl}(R)| \leq 2$. This concludes the proof.

REFERENCES

1. D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990), 1-19.
2. D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Rings between $D[X]$ and $K[X]$* , Houston J. Math **17** (1991), 109-129.
3. N. Bourbaki, "Commutative Algebra," Addison-Wesley, Reading, 1972.
4. A. Zaks, *Half factorial domains*, Israel J. Math. **37** (1980), 281-302.