GD(1) and GD(2) are not preserved in integral closures

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Abstract

In this short note we answer a couple of rather recent questions about the preservation of the GD(1) and GD(2) properties introduced by Malcolmson and Okoh [1]. In particular, it is shown that neither property is preserved in integral closures.

1 Introduction

In [1] the concepts of *GD(1)* and *GD(2)* were introduced. For the sake of completeness, we review them briefly. A ring (commutative with identity) is said to be *GD(1)* if every nonzero element is contained in at most finitely many principal prime ideals. In a certain sense, GD(1) domains may be thought of as generalizations of UFDs.

The concept of a GD(2) ring is a natural "next step." A ring is said to be *GD(2)* if every nonzero element is contained in at most finitely many prime ideals. Clearly, any one-dimensional semi-quasi-local domain is GD(2), but this concept does not generalize UFDs (e.g. any polynomial ring over a field in infinitely many indeterminates is a non-GD(2) UFD).

A natural question for such properties is the question of their stability under specific ring extensions. In particular, the question as to whether the GD(1) and GD(2) properties were preserved under integral closure was brought to the attention of the author. In this paper we provide a negative answer to the question. That is, we will produce examples to show that the GD(1) and GD(2) properties are not preserved under integral closure in general.

In this paper, any ring is commutative with identity, and for a given ring R, we will denote its integral closure by $\overline{R}$.

2 $GD(1)$ Is Not Preserved in Integral Closures

In this section we will produce an example of an integral domain R_0 which is $GD(1)$ (that is, every nonzero element of R_0 lies in only finitely many principal primes), and whose integral closure (which will be denoted R_1) is not $GD(1)$.

Letting K be a field, we begin with the ring:

$$R := K[y, x_1, \cdots, x_n, \cdots, \frac{y}{x_1}, \frac{y}{x_1^2}, \cdots, \frac{y}{x_1^m}, \cdots, \frac{y}{x_n}, \cdots, \frac{y}{x_n^m}, \cdots]$$

(and we note for the sake of clarity that in all stages of this construction, the exponents and subscripts increase without bound).

Letting S be the multiplicatively closed set

$$S = ((x_1) \bigcup (x_2) \bigcup \cdots \bigcup (x_n) \bigcup \cdots)^{\mathrm{c}}$$

(where the superscript "c" denotes the set complement in R) we form the localization

$$R_1 = R_S.$$

To give the relevant example, we consider the subring $R_0 \subseteq R_1$ generated by the elements $y, x_1, x_2^2, x_2^3, x_3^2, x_3^3, \cdots, x_n^2, x_n^3, \cdots$. More precisely, we say

$$R_0 = K[y, x_1, x_2^2, x_2^3, x_3^2, x_3^3, \cdots, x_n^2, x_n^3, \cdots, \frac{y}{x_1}, \frac{y}{x_1^2}, \cdots, \frac{y}{x_2}, \frac{y}{x_2^2}, \cdots, \frac{y}{x_n}, \frac{y}{x_n^2}, \cdots]_T$$

where T is the multiplicatively closed subset of R_0 given by

$$T = ((x_1) \bigcup (x_2^2, x_2^3) \bigcup (x_3^2, x_3^3) \bigcup \cdots \bigcup (x_n^2, x_n^3) \bigcup \cdots)^{\mathrm{c}}$$

(again with the superscript "c" denoting set complement).

We introduce the following lemma that will justify some of the earlier claims.

Lemma 2.1 *The ring, R_0, is a subring of R_1, and the integral closure of R_0 is R_1.*

Proof: To see that $\overline{R_0} = R_1$, it suffices to show that $x_n \in \overline{R_0}$ for all $n \geq 1$ (and that R_0 and R_1 share the same quotient field). Since it is clear that the quotient field of R_0 contains R_1, the fact that they share the same quotient field is apparent. Also note that $x_n^2 \in R_0$, hence, since R_1 is integrally closed, $\overline{R_0} = R_1$. $\diamondsuit$

Theorem 2.2 *The ring R_0 is a GD(1) domain such that the integral closure is not GD(1).*

Proof: To establish the first claim, we must show that R_0 is GD(1). That is, we must establish that every element of R_0 lies in but finitely many principal primes. But by construction, R_0 contains a unique principal prime, namely (x_1). Hence any element is in at most finitely many principal primes.

The fact that the integral closure of R_0 is not GD(1) is also straightforward. Indeed, Lemma 2.1 gives that $\overline{R_0} = R_1$, and in R_1 it is easy to see that the element y is in all the principal primes (x_n), $n \geq 1$. This establishes the theorem. $\diamondsuit$

We remark that this example can easily be adjusted so that the critical element, y, is divisible by any finite number of principal primes whereas in the integral closure, y is divisible by infinitely many primes.

3 $GD(2)$ Is Not Preserved in Integral Closures

In this section, we will show that the associated property (called GD(2)) is also not preserved in integral closures. Again we recall from [1] that the ring R is said to be *GD(2)* if every element of R is contained in only finitely many primes. (In other words, GD(2) is GD(1) with the "principal" restriction removed).

In the previous section we were able to construct an example fairly easily by building a ring in which an element was contained in a unique principal prime. The careful observer will note that in that example "most" elements of the ring were not contained in any principal prime. Of course, in the GD(2) case this analog cannot be achieved since every nonunit is contained in at least one prime ideal. For this reason, our construction in this section is a bit more subtle.

We begin with $\overline{\mathbb{Z}}$, the ring of all algebraic integers. It is well-known (from the work of Nakano, for example [2]), that lying over any fixed rational prime, $p > 0$, there is an infinite number of (height-one) prime ideals. Fixing this prime, p, we consider the subset of the rational integers:

$$S := \{m \in \mathbb{Z} | (m, p) = 1\}$$

and construct the ring

$$R_1 = \overline{\mathbb{Z}}_S.$$

We note that for any nonzero prime ideal $\mathcal{P} \subseteq \overline{\mathbb{Z}}$ lying over $(p) \subseteq \mathbb{Z}$, $\mathcal{P} \bigcap S = \emptyset$. So we see that the ring, R_1, is one-dimensional, integrally closed and possesses infinitely many maximal ideals. Also, since p is contained in every maximal ideal, R_1 fails to be GD(2).

We now construct R_0 by first considering the ideal of R_1 generated by p (pR_1) and letting

$$R_0 = (1, pR_1).$$

That is, R_0 is generated by 1 and the ideal pR_1.

Theorem 3.1 *R_0 is a GD(2) domain whose integral closure is not GD(2).*

Proof: We first claim that $\overline{R_0} = R_1$. To see that the quotient fields coincide, it suffices to show that R_1 is contained in the quotient field of R_0. Letting $r_1 \in R_1$, we note that

$$r_1 = \frac{pr_1}{p}$$

hence r_1 is in the quotient field of R_0.

We next show that $R_0 \supseteq \mathbb{Z}_{(p)} = \mathbb{Z}_S$. Certainly, $\mathbb{Z} \subseteq R_0$, so it suffices to show that for all $m \in S$, $\frac{1}{m} \in R_0$. But since $(m,p) = 1$, we find $a, b \in \mathbb{Z}$ such that

$$am + bp = 1$$

hence

$$a + b(\frac{p}{m}) = \frac{1}{m}.$$

Since $\frac{p}{m} \in pR_1$, we have that $\frac{1}{m} \in R_0$, hence $\mathbb{Z}_S \subseteq R_0$.

Now since $\overline{\mathbb{Z}}_S$ is integral over $\mathbb{Z}_S$, we have that R_1 is integral over R_0, and since R_1 is integrally closed, $R_1 = \overline{R_0}$.

Finally, we claim that R_0 has a unique nonzero prime ideal, pR_1. As R_1, and hence R_0, is one-dimensional it suffices to show that R_0 is quasilocal. To see this we shall show that any nonzero prime ideal of R_1 lies over pR_1. Let $\mathcal{P}$ be an arbitrary nonzero prime ideal of R_1; we will show that $\mathcal{P} \bigcap R_0 = pR_1$. Since the containment $\mathcal{P} \bigcap R_0 \supseteq pR_1$ is clear, we show the other containment.

Assume that $\alpha = n + pr_1 \in \mathcal{P} \bigcap R_0$ with $r_1 \in R_1$ and $n \in \mathbb{Z}$. So we have

$$\alpha - pr_1 = n.$$

Noting that the left hand side of the above equation is in $\mathcal{P}$ and the right hand side is in $\mathbb{Z}$, we see that $n \in \mathcal{P} \bigcap \mathbb{Z} = (p)$. Letting $n = pk$ we have

$$\alpha = pk + pr_1 = p(k + r_1) \in pR_1.$$

Hence we have that the ring R_0 is quasilocal, hence R_0 has a unique nonzero prime ideal. In particular, any nonzero element of R_0 is in finitely many nonzero prime ideals. But in R_1 the element p is contained in infinitely many prime ideals. This concludes the proof. $\diamondsuit$

Remark 3.2 *It is interesting to note that this example is also an example of a quasilocal domain whose integral closure has infinitely many maximal ideals.*

References

[1] P. Malcolmson and F. Okoh. Expansions of prime ideals. Preprint.

[2] N. Nakano. Idealtheorie in einem speziellen unendlichen algebraischen Zahlkorper. *J. Sci. Hiroshima Univ.* **16** (1953), pp. 425–439.