FRAGMENTED DOMAINS HAVE INFINITE KRULL DIMENSION

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Sommaire. On dit qu'un anneau intègre R est fragmenté si pour tout élément non-inversible r de R, il existe un élément non-inversible s de R tel que r$\in \bigcap$Rs^n. On montre, pour un anneau intègre R qui ne soit pas de corps, qu'il existe un idéal maximal de R qui contient une chaîne strictement croissante d'idéaux premiers de R. Si, de plus, R n'ait qu'un nombre fini d'idéaux maximaux, alors on peut faire l'affirmation précédente pour tout ideal maximal de R. Il decoule que tout anneau intègre R qui ne soit pas de corps et qui possède un idéal premier P tel que R+PR_P soit fragmenté doit être de dimension infinie (au sens de Krull). On donne un exemple d'un tel anneau R qui n'est pas fragmenté.

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1. Introduction. Let R be a (commutative integral) domain, with U(R) the group of units of R. As in [14], we say that R is Archimedean if $\bigcap$Rr^n=0 for each r$\in$R$\setminus$U(R). Perhaps the most natural examples of Archimedean domains are Noetherian domains and domains of (Krull) dimension 1: cf. [13, Corollary 1.4]. Contact with these classical types of domains was also achieved in [2, Propositions 3.1 and 3.7(b)] by investigating the impact of having Archimedean overrings. In pursuing the possibility of Ahmes-type expansions of Laurent series, the second-named author came upon a class of non-Archimedean domains in [7, Theorem 2.4]. These "pointwise non-Archimedean" domains are the domains R such that $\bigcap \mathrm{R}\mathrm{r}^n \neq 0$ for each nonzero r∈R\U(R). They are rather plentiful. For example, a valuation domain (more generally, a divided domain, in the sense of [6]) R is pointwise non-Archimedean if and only if each nonzero prime ideal of R has infinite height.

A different class of non-Archimedean domains was introduced in [8] and is our main subject of study in this note. These "fragmented" domains are the domains R for which each r∈R\U(R) has corresponding elements s∈R\U(R) such that $\mathrm{r} \in \bigcap \mathrm{R}\mathrm{s}^n$. According to [8, Corollary 2.6], a valuation domain (R,$\mathcal{M}$) is fragmented if and only if $\mathcal{M}$ is unbranched in R, in the sense of [10]. It is known that if a fragmented non-field domain R is either quasilocal or semiquasilocal and treed, then R is infinite-dimensional [8, Corollary 2.8 and Proposition 2.9]. We find this remarkable, in view of the behavior of the recently introduced class of "antimatter" domains. Following [5], a domain R is called an antimatter domain if R has no irreducible elements. It is evident that each fragmented domain is antimatter, but the converse is false. Indeed, although a fragmented non-field valuation domain must be infinite-dimensional, there exist two-dimensional antimatter valuation domains. Given such a domain R, examples show that that the one-dimensional overring of R may [5, Example 2.6(b)] or may not [5, Example 2.6(a)] be antimatter.

In light of the above facts, our main purpose here is to show that fragmented domains are indeed very special antimatter domains. To this end, Theorem 2.3 establishes that any fragmented non-field domain is infinite-dimensional. Moreover, Corollary 2.5 yields that if a domain R has a fragmented CPI-overring (in the sense of [3]), then R is infinite-dimensional; Example 2.7 shows that such an R need not itself be fragmented. Finally, Proposition 2.9 should be contrasted with the "antimatter" behavior in [5, Example 2.6(b)] mentioned above.

As usual, $\mathrm{ht_D}(\mathcal{Q})$ denotes the height of a prime ideal $\mathcal{Q}$ in a domain D, dim denotes Krull dimension, and $\subset$ and $\supset$ denote proper inclusions. Any unexplained material is standard, as in [10], [11].

2. Results. Before stating our main result, Theorem 2.3, we give two lemmas. The context for all three of these results is a fragmented domain R which is not a field. Our strategy is to produce a strictly increasing chain of prime ideals of R by developing a corresponding decreasing chain of multiplicatively closed subsets of R. (This strategy is motivated by Arnold's method of proving infinite-dimensionality for certain formal power series rings in [1, Theorem 1].) To this end, we proceed to define a useful sequence $\{x_n\}$ of nonunits of R.

The construction begins by choosing any nonzero element $x_1 \in \mathrm{R}\backslash\mathrm{U(R)}$; it is, of course, possible to do this since R is assumed not to be a field. Next, since R is assumed fragmented, we may inductively produce a sequence $\{x_n\}$ of nonzero nonunits of R such that

$$x_n \in \bigcap_{k=1}^{\infty} \mathrm{R}x_{n+1}^k \text{ for each } n \geq 1.$$

Observe that $\mathrm{R}x_n \subset \mathrm{R}x_{n+1}$ for each n, since $x_n \in \mathrm{R}x_{n+1}^2$ and x_{n+1} is a nonunit of R. Also, $x_n \in \mathrm{R}x_m$ for any positive integers $n \leq m$.

Consider the set $S = \{y \in \mathrm{R} | y = 1 + rx_n$ for some $r \in$R and $n \geq 1\}$. It follows easily from the above remark that S is a multiplicatively closed subset of R. We come now to the crux of the construction. For each $n \geq 2$, let S_n be the multiplicative subset of R generated by $\{x_m | m \geq n\} \bigcup S$; that is:

$$S_n = \{y \in \mathrm{R} | y = x_n^{k_n} ... x_{n+p}^{k_{n+p}} s, \text{forsome } s \in S, p \geq 0, \text{with } k_i \geq 0 \text{ forall } i\}.$$

Observe that $S_2 \supseteq S_3 \supseteq ... \supseteq S_n \supseteq S_{n+1} \supseteq ... \supset S$; indeed, since x_n is a nonunit, it is easy to verify that $x_n \in S_n \setminus S$ for each $n \geq 2$.

Lemma 2.1. Under the above hypotheses and notation, if $n \geq 1, r \in$R and $s_{n+1} \in S_{n+1}$, then $rx_n + s_{n+1} \in S_{n+1}$.

Proof. Fix a description $s_{n+1} = x_{n+1}^{k_{n+1}} ... x_{n+p}^{k_{n+p}} (1 + bx_m)$, for some $p \geq 0$, $b \in$R, and $m \geq 1$, with each $k_i \geq 0$. Note that p may be taken to be any positive integer, and, without loss of generality, we may then replace m so as to ensure that $m \geq n + p$ (using the observation from the next-to-last paragraph before Lemma 2.1). We shall find several ways to rewrite

$$z = rx_n + s_{n+1} = rx_n + x_{n+1}^{k_{n+1}} ... x_{n+p}^{k_{n+p}} (1 + bx_m). \qquad (*)$$

First, since $x_n \in \bigcap \mathrm{R}x_{n+1}^j$, we can write

$$x_n = r_n x_{n+1}^{k_{n+1}+1} = r_n x_{n+1}^{k_{n+1}} x_{n+1} \text{ for some } r_n \in \mathrm{R}. \qquad (1)$$

Similarly, as $x_{n+1} \in \bigcap \mathrm{R}x_{n+2}^j$, we have

$$x_{n+1} = r_{n+1} x_{n+2}^{k_{n+2}+1} = r_{n+1} x_{n+2}^{k_{n+2}} x_{n+2} \text{ for some } r_{n+1} \in \mathrm{R}. \qquad (2)$$

Proceeding in this fashion, we produce expressions for x_{n+i}, for all $0 \leq i \leq p - 1$, of the form

$$x_{n+i} = r_{n+i} x_{n+i+1}^{k_{n+i+1}} x_{n+i+1} \text{ for some } r_{n+i} \in \mathrm{R}.$$

By repeatedly rewriting the rx_n term in z, we see via (*), (1), (2), ... that

$$z = rr_n x_{n+1}^{k_{n+1}} x_{n+1} + x_{n+1}^{k_{n+1}} ... x_{n+p}^{k_{n+p}} (1 + bx_m)$$

$$= rr_n x_{n+1}^{k_{n+1}} r_{n+1} x_{n+2}^{k_{n+2}} x_{n+2} + x_{n+1}^{k_{n+1}} ... x_{n+p}^{k_{n+p}} (1 + bx_m) =$$

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$$= rr_n x_{n+1}^{k_{n+1}} r_{n+1} x_{n+2}^{k_{n+2}} ... r_{n+p-1} x_{n+p}^{k_{n+p}} x_{n+p} + x_{n+1}^{k_{n+1}} ... x_{n+p}^{k_{n+p}} (1 + bx_m).$$

Hence, we conclude that

$$z = x_{n+1}^{k_{n+1}} ... x_{n+p}^{k_{n+p}} (r' x_{n+p} + (1 + bx_m))$$

where $r' = rr_n r_{n+1} ... r_{n+p-1} \in \mathrm{R}$.

Since $n + p \leq m$, we have $x_{n+p} = r'' x_m$ for some $r'' \in \mathrm{R}$. It follows that

$$rx_n + s_{n+1} = z = x_{n+1}^{k_{n+1}} ... x_{n+p}^{k_{n+p}} s$$

where $s := 1 + (r'r'' + b)x_m \in S$, whence $z \in S_{n+1}$. $\diamondsuit$

Lemma 2.2. Under the above hypotheses and notation, $\mathrm{R}x_1 \bigcap S_2 = \phi$.

Proof. Deny. Then $rx_1 \in S_2$ for some $r \in$R. Fix a description

$$rx_1 = x_2^{k_2}...x_n^{k_n}(1 + \overline{r}x_m)$$

for some $n \geq 2$, $\overline{r} \in$R, and $m \geq 1$, with each $k_i \geq 0$. Without loss of generality, $m \geq n$. As in the proof of Lemma 2.1, the choice of the sequence $\{x_1, x_2, ...\}$ leads to the $n-1$ equations

$$x_1 = r_1 x_2^{k_2+1} = r_1 x_2^{k_2} x_2$$

$$x_2 = r_2 x_3^{k_3+1} = r_2 x_3^{k_3} x_3$$

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$$x_{n-1} = r_{n-1} x_n^{k_n+1} = r_{n-1} x_n^{k_n} x_n$$

for some $r_1, r_2, ..., r_{n-1} \in$R. Combining the n displayed equations, we have

$$x_2^{k_2}...x_n^{k_n}(1 + \overline{r}x_m) = rx_1 = rr_1 x_2^{k_2} x_2 = rr_1 x_2^{k_2} r_2 x_3^{k_3} x_3 = ...$$

$$= rr_1 x_2^{k_2} r_2 x_3^{k_3} ... r_{n-1} x_n^{k_n} x_n = x_2^{k_2} ... x_n^{k_n} rr_1 ... r_{n-1} x_n.$$

By cancellation, $1+\overline{r}x_m = rr_1...r_{n-1}x_n = r'x_n$, with $r' := rr_1...r_{n-1}$. As $n \leq m$, $x_n = r''x_m$

for some $r'' \in$R. Thus $1+\overline{r}x_m = r'r''x_m$, whence $1 = (r'r''-\overline{r})x_m \in \mathrm{R}x_m$, contradicting $x_m \in$R$\setminus$U(R). $\diamondsuit$

Theorem 2.3. Let R be a fragmented domain which is not a field, and let x be a nonzero nonunit of R. Then some maximal ideal $\mathcal{M}$ of R contains a strictly increasing chain $\{\mathcal{P}_k\}$ of prime ideals of R such that $x \in \mathcal{P}_k$ for each k. In particular, $\mathrm{ht}_{\mathrm{R}}(\mathcal{M})$=$\infty$ and dim(R)=∞.

Proof. As above, construct a sequence $\{x_n\}$ of nonzero nonunits of R such that $x_n \in \bigcap_{k=1}^{\infty} \mathrm{R}x_{n+1}^k$ for each $n \geq 1$, with $x_1 = x$. We shall produce $\mathcal{M}$ and $\{\mathcal{P}_k\}$ as asserted, with the additional feature that $x_n \in \mathcal{P}_n$ for each $n \geq 1$, one upshot being that $\mathcal{M}$ contains $\{x_n\}$. To begin, combine Lemma 2.2 and [11, Theorem 1] to produce $\mathcal{P}_1 \in$Spec(R) such that $\mathrm{R}x_1 \subseteq \mathcal{P}_1$ and $\mathcal{P}_1 \bigcap S_2 = \phi$. (Recall that the sequence $\{S_n | n \geq 2\}$ was defined prior to the statement of Lemma 2.1.)

We proceed inductively. Suppose that $n \geq 1$ and

$$x_1 \in \mathcal{P}_1 \subset \mathcal{P}_2 \subset ... \subset \mathcal{P}_n$$

for prime ideals $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n$ of R such that $\mathcal{P}_i \bigcap S_{i+1} = \phi$ for all $1 \leq i \leq n$. We shall produce a prime ideal $\mathcal{P}_{n+1}$ of R which properly contains $\mathcal{P}_n$. Consider the ideal $I_{n+1} := \mathrm{R}x_{n+1} + \mathcal{P}_n$. Observe that I_{n+1} properly contains $\mathcal{P}_n$; indeed, $x_{n+1} \in I_{n+1} \setminus \mathcal{P}_n$ since $x_{n+1} \in S_{n+1}$. It suffices to show that $I_{n+1} \bigcap S_{n+2} = \phi$. Indeed, given this claim, an appeal to [11, Theorem 1] produces $\mathcal{P}_{n+1} \in$Spec(R) such that $I_{n+1} \subseteq \mathcal{P}_{n+1}$ and $\mathcal{P}_{n+1} \bigcap S_{n+2} = \phi$; in particular, $\mathcal{P}_n \subset I_{n+1} \subseteq \mathcal{P}_{n+1}$. With the strictly increasing chain $\{\mathcal{P}_k\}$ thus inductively constructed, we need only take $\mathcal{M}$ to be any maximal ideal of R which contains the proper ideal $\bigcup \mathcal{P}_k$ (cf. [11, Theorem 9]).

It remains only to prove the claim that $I_{n+1} \bigcap S_{n+2} = \phi$. Deny. Then $rx_{n+1} + p_n = s_{n+2} \in S_{n+2}$ for some $r \in$R, $p_n \in \mathcal{P}_n$, and $s_{n+2} \in S_{n+2}$. Hence, $p_n = (-r)x_{n+1} + s_{n+2}$, which, by Lemma 2.1, is an element of S_{n+2}. As $S_{n+2} \subseteq S_{n+1}$, we have $p_n \in S_{n+1}$, whence $p_n \in \mathcal{P}_n \bigcap S_{n+1} = \phi$, the desired contradiction. $\diamondsuit$

The next result generalizes the treed case given in [8, Proposition 2.9].

Corollary 2.4. Let R be a semiquasilocal fragmented domain which is not a field, and let $\mathcal{M}$ be a maximal ideal of R. Then $\text{ht}_{\text{R}}(\mathcal{M})=\infty$; in fact, $\mathcal{M}$ contains a strictly increasing chain of prime ideals of R.

Proof. Let $\mathcal{M} = \mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n$ be the (pairwise distinct) maximal ideals of R. As the quasilocal case was established in [8, Corollary 2.8], we may suppose that $n > 1$. By the prime avoidance lemma (cf. [11, Theorem 81]), we may choose $x_1 \in \mathcal{M} \setminus (\mathcal{M}_2 \bigcup ... \bigcup \mathcal{M}_n)$. Use the "fragmented" hypothesis as above to construct a sequence $x_1, x_2, ...$ of nonzero nonunits of R such that $x_j \in \bigcap_{k=1}^{\infty} \text{R}x_{j+1}^k$ for each $j \geq 1$. As in the proof of Theorem 2.3, there exists a strictly increasing chain $\{\mathcal{P}_k\}$ of prime ideals of R and an index j, $1 \leq j \leq n$, such that

$$x_1 \in \mathcal{P}_1 \subset \mathcal{P}_2 \subset ... \subset \mathcal{P}_k \subset ... \subset \mathcal{M}_j.$$

By the choice of x_1, we have $j = 1$, and all the assertions follow. $\diamondsuit$

We show in Corollary 2.5 that a domain R must be infinite-dimensional in case a certain type of overring of R is fragmented. Recall first from [3], that if $\mathcal{P}$ is a prime ideal of a domain R, then the *CPI-extension of R with repect to* $\mathcal{P}$ is $\mathrm{R}+\mathcal{P}\mathrm{R}_{\mathcal{P}}$, which may, of course, be viewed as the pullback $\mathrm{R}_{\mathcal{P}} \times_{\mathrm{R}_{\mathcal{P}}/\mathcal{P}\mathrm{R}_{\mathcal{P}}} \mathrm{R}/\mathcal{P}$. As a partially ordered set, $\mathrm{Spec}(\mathrm{R}+\mathcal{P}\mathrm{R}_{\mathcal{P}})$ is (order-) isomorphic to $\{\mathcal{Q} \in \mathrm{Spec}(\mathrm{R})|$ either $\mathcal{Q} \subseteq \mathcal{P}$ or $\mathcal{P} \subseteq \mathcal{Q}\}$ by the standard contraction map $\Gamma \longmapsto \Gamma \bigcap \mathrm{R}$,: see [3, Corollary 2.8] and [9, Proposition 4.1].

Corollary 2.5. Let R be a domain and $\mathcal{P}$ a nonzero prime ideal of R such that $\mathrm{R}+\mathcal{P}\mathrm{R}_{\mathcal{P}}$, the CPI-extension of R with repect to $\mathcal{P}$, is a fragmented domain. Then some maximal ideal $\mathcal{M}$ of R which contains $\mathcal{P}$ also contains a strictly increasing chain $\{\mathcal{P}_k\}$ of prime ideals of R such that for each k, either $\mathcal{P}_k \subseteq \mathcal{P}$ or $\mathcal{P} \subseteq \mathcal{P}_k$. In particular, $\mathrm{ht}_{\mathrm{R}}(\mathcal{M})=\infty$ and $\dim(\mathrm{R})=\infty$.

Proof. $\mathcal{P}\mathrm{R}_{\mathcal{P}}$ is the (only) prime ideal of $\mathrm{R}+\mathcal{P}\mathrm{R}_{\mathcal{P}}$ which contracts to $\mathcal{P}$. As $\mathcal{P}\mathrm{R}_{\mathcal{P}} \neq 0$, Theorem 2.3 yields a strictly increasing chain $\{\Gamma_k\}$ of prime ideals of $\mathrm{R}+\mathcal{P}\mathrm{R}_{\mathcal{P}}$. By the order-isomorphism noted above, $\{\mathcal{P}_k := \Gamma_k \bigcap \mathrm{R}\}$ is a strictly increasing chain of prime ideals of R and for each k, either $\mathcal{P}_k \subseteq \mathcal{P}$ or $\mathcal{P} \subseteq \mathcal{P}_k$. It now suffices to take $\mathcal{M}$ to be any maximal ideal of R which contains $\bigcup \mathcal{P}_k \bigcup \mathcal{P}$. $\diamondsuit$

Example 2.7 shows that a domain R satisfying the hypotheses of Corollary 2.5 need not be fragmented. First, we need the following result, which is of some independent interest.

Proposition 2.6. Let R be a fragmented domain with exactly two maximal ideals, say $\mathcal{M}$ and $\mathcal{N}$. Then neither $\mathcal{M}$ nor $\mathcal{N}$ has height 1 in R.

Proof. The assertion follows immediately from Corollary 2.4. We next give an alternate, direct proof.

Deny. Without loss of generality, $\mathrm{ht_R}(\mathcal{N})$=1. Since $\dim(\mathrm{R}_{\mathcal{N}})$=1, $\mathrm{R}_{\mathcal{N}}$ is Archimedean [13, Corollary 1.4]. Choose $r \in \mathcal{N} \setminus \mathcal{M}$. As R is fragmented, $r \in \bigcap \mathrm{R}s^n$ for some $s \in$R\U(R)=$\mathcal{M} \bigcup \mathcal{N}$. Since $\mathrm{R}_{\mathcal{N}}$ is Archimedean, $s \notin \mathcal{N}\mathrm{R}_{\mathcal{N}}$. Hence $s \notin \mathcal{N}$, and so $s \in \mathcal{M}$. Then $r \in \mathrm{R}s \subseteq \mathcal{M}$, the desired contradiction. $\diamondsuit$

Example 2.7. By verifying the conditions in [12, Corollary 3.6], one verifies the existence of a valuation domain $(V, \mathcal{M})$ such that Spec(V), as a partially ordered set, is order-isomorphic to

$$-\infty < ... < -n < ... < -2 < -1 < 1 < 2 < ... < n < ... < \infty.$$

(It is interesting to note via [8, Theorem 2.5] and [7, Theorem 2.6(ii)] that V is both fragmented and pointwise non-Archimedean.) Let X be an indeterminate over V. Set $W_1 := V[X]_{XV[X]}, W_2 := V[X]_{\mathcal{M}[X]}$, and R :=$W_1 \bigcap W_2$. Then R is a non-fragmented domain which satifies the hypotheses of Corollary 2.5.

For a proof, observe first that $V[X]$ is a GCD-domain (cf. [11, Exercise 9, p. 42]) and hence, since they are localizations of a GCD-domain, W_1 and W_2 are GCD-domains [4, Exercise 21, p. 551]. Moreover, W_1 and W_2 are each quasilocal treed domains; i.e., Spec(W_i) is linearly ordered by inclusion, $i = 1, 2$. (This is evident if $i = 1$, for $XV[X]$ is an upper of 0, whence $\dim(W_1)$=$\mathrm{ht}_{V[X]}(XV[X])$=1: cf. [11, Theorem 37]. For $i = 2$, the fact that V is a Prüfer domain ensures that $\{\mathcal{Q} \in \mathrm{Spec}(V[X]) | \mathcal{Q} \subseteq \mathcal{M}[X]\} \longrightarrow \mathrm{Spec}(V)$, $(\mathcal{Q} \longmapsto \mathcal{Q} \bigcap V)$, is an order-isomorphism: cf. [10, Theorem 19.15]. Therefore, W_2 inherits the "quasilocal treed" property from V.) It follows that W_1 and W_2 are valuation domains, since they are each quasilocal treed GCD-domains: cf. [15, Theorem 3.7]. By considering prime spectra, we see that neither W_1 nor W_2 is a localization of the other, and so neither W_1 nor W_2 contains the other. It follows that R is a Bézout domain with exactly two maximal ideals, and that these maximal ideals may be labeled $\mathcal{M}_1$ and $\mathcal{M}_2$ so that $\mathrm{R}_{\mathcal{M}_i} = W_i, i = 1, 2$ [11, Theorem 107]. By a standard criterion for branchedness in a Prüfer domain [10, Theorem 23.3(e)], we see that $\mathcal{M}_1$ is branched in R and $\mathcal{M}_2$ is unbranched in R; indeed, $\mathcal{M}_1$ (resp., $\mathcal{M}_2$) is not (resp., is) the union of the set of prime ideals of R which are properly contained in it. Also by Proposition 2.6, R is not a fragmented domain, since $\mathrm{ht}_{\mathrm{R}}(\mathcal{M}_1)$=dim$(W_1)$=1.

It suffices to show that if $\mathcal{P}$ is any nonzero prime ideal of R such that $\mathcal{P} \subseteq \mathcal{M}_2$, then T:=R+$\mathcal{P}\mathrm{R}_{\mathcal{P}}$ is a fragmented domain. Since T is an overring of the Bézout domain R, it follows that T is also a Bézout domain. However, Spec(T) is order-isomorphic to $\{\mathcal{Q} \in$ Spec(R)$|$ either $\mathcal{Q} \subseteq \mathcal{P}$ or $\mathcal{P} \subseteq \mathcal{Q}\}$=Spec(R)$\setminus\{\mathcal{M}_1\}$ by [3, Corollary 2.8], and so T is quasilocal. Therefore, T is a valuation overring of R whose maximal ideal, $\mathcal{M}_2 + \mathcal{P}\mathrm{R}_{\mathcal{P}}$, meets R in $\mathcal{M}_2$. As R is a Prüfer domain, T=$\mathrm{R}_{\mathcal{M}_2}$; i.e., T=$W_2$. Finally, W_2 is fragmented by [8, Corollary 2.6] since $\mathcal{M}_2 W_2$ is unbranched in W_2. $\diamondsuit$

The interested reader may verify that the proof given for Example 2.7 may be adapted to any Bézout domain R with exactly two maximal ideals, say $\mathcal{M}_1$ and $\mathcal{M}_2$, such that $\mathrm{ht}_{\mathrm{R}}(\mathcal{M}_1)$=1 and $\mathcal{M}_2$ is unbranched in R. (Then, as above, one proves that R satisfies the hypotheses of Corollary 2.5 but is not fragmented.) Apart from the explicit construction given in Example 2.7, additional examples of such domains R are available via [12, Theorem 3.1].

Remark 2.8. The proof of Example 2.7 shows that $\mathcal{M}_1$ is the *only* nonzero prime ideal $\mathcal{Q}$ of R such that R+$\mathcal{Q}$R$_\mathcal{Q}$ is *not* fragmented. In view of Corollary 2.5, one might expect that it would be easy to prove directly that a non-field domain D is infinite-dimensional if D+$\mathcal{Q}$D$_\mathcal{Q}$ is fragmented for *each* nonzero $\mathcal{Q}$ ∈Spec(D). In fact, this is indeed easy, for if $\mathcal{Q}$ is a maximal ideal of D, then D+$\mathcal{Q}$D$_\mathcal{Q}$ = D$_\mathcal{Q}$ and an appeal to [8, Corollary 2.8] (or Corollary 2.4) suffices. For this reason, we next seek a stronger result along these lines.

Specifically, we raise the following question. If R is a domain such that R+$\mathcal{P}$R$_\mathcal{P}$ is fragmented for each nonzero nonmaximal ideal $\mathcal{P}$ ∈Spec(R), and if $\mathcal{M}$ is a maximal ideal of R, must it be the case that either ht$_\mathrm{R}$($\mathcal{M}$)≤ 1 or ht$_\mathrm{R}$($\mathcal{M}$)=∞? (Note, in contrast to Corollary 2.5, that the first alternative is viable, since any domain R such that dim(R)≤ 1 vacuously satisfies the above hypotheses.) It follows from Corollary 2.4 (and [3, Corollary 2.8]) that the answer is affirmative if each nonzero prime ideal of R is contained in only finitely many maximal ideals of R. ♢

Next, we answer the "proper overring" analogue of the question raised in Remark 2.8, at least in the quasilocal treed case. To put Proposition 2.9 into perspective, recall from [5, Example 2.6 (b)] that there exists a two-dimensional valuation domain R such that each overring of R fails to have an irreducible element. One consequence of Proposition 2.3 is that if R is such a valuation domain and P is its height 1 prime ideal, then R$_\mathrm{P}$ is not a fragmented domain. Of course the same conclusion also follows from Proposition 2.9.

Propostion 2.9. For a domain R, the following conditions are equivalent:

(1) R is a quasilocal treed domain and each proper overring is a fragmented domain;

(2) R is a valuation domain and dim(R)≤ 1.

Proof. Since fields are fragmented, (2)$\Longrightarrow$(1) trivially (cf. [11, Exercise 29, p. 43]). For the converse, suppose (1), and let $\mathcal{M}$ denote the maximal ideal of R. Without loss of generality, R is not a field; that is, $\mathcal{M} \neq 0$. Suppose that $\mathrm{ht}_{\mathrm{R}}(\mathcal{M}) < \infty$. Then there exists a valuation overring V of R and $\mathcal{Q} \in$Spec(V) such that $\mathcal{P} := \mathcal{Q} \bigcap \mathrm{V}$ is a height-one prime ideal of R (cf. [10, Corollary 19.7(1)]). Let Γ denote the pseudo-radical of V, that is, the intersection of all the nonzero prime ideals Γ_α of V. As V is quasilocal treed, we may restrict attention to those Γ_α contained in $\mathcal{Q}$. Any such Γ_α meets R in $\mathcal{P}$, and hence so does Γ. Thus $\Gamma \neq 0$, and so $\mathrm{ht}_{\mathrm{V}}(\Gamma)=1$. As V_Γ is a quasilocal one-dimensional overring of R, it follows from [8, Corollary 2.8] that V_Γ is not fragmented. Therefore, by (1), $\mathrm{R}=\mathrm{V}_\Gamma$, a one-dimensional valuation domain.

It remains only to show that the case $\mathrm{ht}_{\mathrm{R}}(\mathcal{M})=\infty$ leads to a contradiction. Let $\mathcal{N}$ be any nonzero nonmaximal prime ideal of R. Use [11, Theorem 11] to produce adjacent prime ideals of $\mathrm{R}_{\mathcal{N}}$, and by intersecting them with R, obtain adjacent prime ideals $\mathcal{P}_1 \subset \mathcal{P}_2$ of R with $\mathcal{P}_2 \subseteq \mathcal{N}$. Since $\mathrm{T}:=\mathrm{R}_{\mathcal{P}_2}$ is then a proper overring of R, (1) yields that T is fragmented. However, T is quasilocal treed and if we choose $r \in \mathcal{P}_2 \setminus \mathcal{P}_1$, "adjacency" ensures that the maximal ideal of T is the radical of Tr. This contradiction to [8, Proposition 2.4] completes the proof. $\diamondsuit$

In closing, we show that, at least for divided domains, Theorem 2.3 has pinpointed the key feature determining whether a given domain is fragmented.

Corollary 2.10. Let (R, $\mathcal{M}$) be a divided domain. Then the following conditions are equivalent:

(1) For each nonzero $x \in \mathcal{M}$, there exists a strictly increasing chain of prime ideals of

R which contain x;

(2) R is a fragmented domain.

Proof. Without loss of generality, R is not a field, Theorem 2.3 yields that (2)$\Longrightarrow$(1). Conversely, since (R, $\mathcal{M}$) is divided, [8, Theorem 2.5] shows that (2) holds if (and only if) $\mathcal{M}$ is the union of the nonmaximal prime ideals of R. As this condition is evidently a consequence of (1), we also have that (1)$\Longrightarrow$(2). $\diamondsuit$

REFERENCES

[1] J. T. Arnold, *Krull dimension in power series rings*, Trans. Amer. Math. Soc. **177**(1973), 299-304.

[2] V. Barucci and D. E. Dobbs, *On chain conditions in integral domains*, Canad. Math. Bull. **27**(1984), 351-359.

[3] M. B. Boisen, Jr. and P. B. Sheldon, *CPI-extensions: overrings of integral domains with special prime spectrums*, Can. J. Math. **29**(1977), 722-737.

[4] N. Bourbaki, *Commutative Algebra*, Addison-Wesley, Reading, 1972.

[5] J. Coykendall, D. E. Dobbs, and B. Mullins, *On integral domains with no atoms*, submitted for publication.

[6] D. E. Dobbs, *On locally divided integral domains and CPI-overrings*, Internat. J. Math. Math. Sci. **4**(1981), 119-135.

[7] D. E. Dobbs, *Ahmes expansions of formal Laurent series and a class of non-Archimedean*

integral domains, J. Algebra **103**(1986), 193-201.

[8] D. E. Dobbs, *Fragmented integral domains*, Port. Math. **43**(1985-1986), 463-473.

[9] M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. **123**(1980), 331-355.

[10] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

[11] I. Kaplansky, *Commutative Rings*, rev. ed., Univ. Chicago Press, Chicago, 1974.

[12] W. J. Lewis, *The spectrum of a ring as a partially ordered set*, J. Algebra **25**(1973), 419-434.

[13] J. Ohm, *Some counterexamples related to integral closure in D[[x]]*, Trans. Amer. Math. Soc. **122**(1966), 321-333.

[14] P. B. Sheldon, *How changing D[[x]] changes its quotient field*, Trans. Amer. Math. Soc. **159**(1971), 223-244.

[15] P. B. Sheldon, *Prime ideals in GCD-domains*, Can. J. Math. **26**(1974), 98-107.

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