MATH 793
SUMMER 2003
HOMEWORK 2

Due Monday July 14, 2003.

1. Let R be a domain and $S \subseteq R$ be a multiplicatively closed set ($0 \notin S$). Prove the following.

a) (5 pt) If R is a UFD then so is R_S.
b) (5 pt) If R is a PID then so is R_S.
c) (5 pt) If R is Noetherian then so is R_S.
d) (5 pt) $\dim(R_S) \leq \dim(R)$.

2. Let R be a domain with quotient field K. An element $\omega \in K$ is called *integral* if ω is a root of some monic polynomial $f(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 \in R[x]$. The set of all elements integral over R forms a ring extension called the integral closure of R, and any ring that coincides with its integral closure is called *integrally closed*.

a) (5 pt) Prove that any valuation domain is integrally closed.
b) (5 pt) Prove that any UFD is integrally closed.

3. Let R be a domain with quotient field K. An element $\omega \in K$ is called *almost integral* if there exists a nonzero $r \in R$ such that $r\omega^n \in R$ for all $n \geq 0$. The set of all elements almost integral over R forms a ring extension called the complete integral closure of R, and any ring that coincides with its complete integral closure is called *completely integrally closed*.

a) (5 pt) Show that any element that is integral is almost integral.
b) (5 pt) Prove that any UFD is completely integrally closed.
c) (5 pt) Give an example of an almost integral element that is not integral (remember to specify the domains where you are working).
d) (5 pt) If V is a valuation domain, show that V is completely integrally closed if and only if $\dim(V) \leq 1$.
e) (5 pt) If V is a valuation domain, compute the complete integral closure of V.