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Research Statement

My interests lie primarily with non-Noetherian rings, particularly Prüfer domains. A *Prüfer domain* can be defined as an integral domain in which every non-zero fractional ideal is invertible (making it a non-Noetherian version of a Dedekind domain), or as an integral domain for which, locally at every maximal ideal, the ideals are totally ordered by inclusion (making it a global version of a valuation domain), or as an integral domain for which the ideal lattice is distributive (called a "Multiplikationsring" by Krull). These are only three of over 150 known equivalent definitions for a Prüfer domain.

My approach to the study of Prüfer domains has been by means of conductor squares. For commutative rings $R \subseteq T$, the *conductor* of T into R is the largest common ideal C of R and T; we assume always that C is non-zero. Setting $A = R/C$ and $B = T/C$, we obtain the natural surjections $\eta_1 : T \twoheadrightarrow B$ and $\eta_2 : R \twoheadrightarrow A$ and the inclusions $\iota_1 : A \hookrightarrow B$ and $\iota_2 : R \hookrightarrow T$. These maps yield a commutative diagram, called a *conductor square*, which defines R as a pullback of η_1 and ι_1.

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & T \arrow[d] \\ A \arrow[r, hook] & B \end{tikzcd} \tag{$\square$}$$

That is, $R = \eta_1^{-1}(\iota_1(A))$. If we start with a ring surjection $\eta_1 : T \twoheadrightarrow B$ and an inclusion of rings $\iota_1 : A \hookrightarrow B$, then the pullback defines a subring R of T with conductor $C = \ker \eta_1$ and a conductor square ($\square$). We study the extent to which properties of A and T are "inherited" by R.

Let D be an integral domain with field of fractions K, and let R be any Prüfer domain between the polynomial rings $D[X]$ and $T = K[X]$, defined by the conductor square ($\square$). In my dissertation [1], I showed that there exist distinct, irreducible polynomials $F_1, ..., F_r \in K[X]$, such that the conductor of T into R is $K[X] \cdot F_1 \cdot ... \cdot F_r$. We say that such a ring R *is defined by a conductor square of type* ($\boxtimes$) if the polynomials $F_1, ..., F_r$ are all monic with coefficients in D.

I plan to work on the following questions during the next few years.

1. In [9], Glaz describes three ways of extending definition of a Prüfer domain to any commutative ring with zero-divisors. Once the domain assumption is dropped, many of the defining properties of a Prüfer domain are no longer equivalent. A *chained ring* is a ring in which the ideals are totally ordered under inclusion; an *arithmetical ring* is a ring which, locally at every maximal ideal, is a chained ring. In my dissertation [1], necessary and sufficient conditions are given such that a ring R defined by a conductor square ($\square$) is an arithmetical ring. Do similar results hold for other extended definitions? A *Prüfer ring* is a ring in which all finitely generated, regular ideals are invertible;

a *semi-hereditary ring* is a ring in which all finitely generated ideals are projective. Can one give necessary and sufficient conditions, similar to those given for arithmetical rings, that R be a Prüfer ring or a semi-hereditary ring?

2. Let $\Re$ be a commutative ring. A finitely generated ideal I is said to be *n-generated* if, for some positive integer n, there exist elements $a_1, ..., a_n \in \Re$ such that $I = \Re a_1 + ... + \Re a_n$. $\Re$ is said to have the *n-generator property* if all of its finitely generated ideals are n-generated. Now, let T be an integral domain with subfield K and maximal ideal M such that $T = K + M$. For any subring $D \subseteq K$, we refer to subring $D + M \subseteq T$ as a *$D + M$ construction*. The ring $R = D + M$ is defined by a conductor square ($\square$), with conductor $C = M$ and quotient $A = R/M = D$ (the classical example being $D + XK[X]$). In [3], Brewer and Rutter show that R has the n-generator property if and only if the same is true for D and $T = K + M$. In the joint work [2], we show that the ring $\mathrm{Int}(E, D) = \{g \in K[X] : g(E) \subset D\}$, *the ring of integer-valued polynomials determined by a subset* $E \subset K$, has the n-generator property if and only if D does, provided that E is finite and non-empty, and $n \geq 2$. Under what conditions can one prove a similar result for a general ring R defined by the conductor square ($\square$)?

3. Let S be any Prüfer domain between $D[X]$ and $K[X]$. If it is the case that there is a non-zero conductor of $K[X]$ into S, then by [1], $S \simeq R$ where R is a Prüfer domain defined by a conductor square of type ($\boxtimes$). In [11], Zafrullah says that there are basically two such rings: (1) $A + XB[X] = \{a_0 + \sum_{i=1}^{n} b_i X^i : a_0 \in A, b_i \in B\}$ where $D \subseteq A \subseteq B \subseteq K$ are ring extensions, and (2) the intermediate rings not of this form. He goes on to say that the only well-studied ring of type (2) is the ring $\mathrm{Int}(E, D)$. Perhaps the nicest example of a ring of type (1) is the classical $D + XK[X]$ construction. It is not hard to see that $D + XK[X]$ is defined by a conductor square of the form ($\boxtimes$). If E happens to be a finite subset of D, then [10] together with [1] show that the the ring $\mathrm{Int}(E, D)$ is also defined by a conductor square of type ($\boxtimes$). In [5], Chapman, Loper and Smith express the class group $\mathfrak{C}(\mathrm{Int}(E, D))$ in terms of the class group $\mathfrak{C}(D)$. Similarly, [3] shows that, if D is a Prüfer domain, then $\mathfrak{C}(K + M) \simeq \mathfrak{C}(D + M)/\mathfrak{C}(D)$ so that that $\mathfrak{C}(D + XK[X]) = \mathfrak{C}(D)$. Can one prove a similar result for a general ring R defined by a conductor square of type ($\boxtimes$)? What about the general conductor square ($\square$)? In [11], Zafrullah relates the Krull dimension $\dim(A + XB[X])$ to the Krull dimensions $\dim(A)$ and $\dim(B[X])$, and in [3], Brewer and Rutter show that $\dim(D + M) = \max\{\mathrm{height}_T(M) + \dim(D), \dim(T)\}$. Are there similar results for a general ring R defined by a conductor square of type ($\boxtimes$)? What about the general conductor square ($\square$)? Finally, are there any conditions that one can impose on ($\boxtimes$) to ensure that R is a ring of type (1)?

4. Let $\Re$ be any commutative ring and let I be an ideal of $\Re$. We say that I is *strongly n-generated* if it is n-generated and one of the n elements can be chosen at random from the non-zero elements of I. We say that $\Re$ has the

strong n-generator property if all of its finitely generated ideals are strongly n-generated. In [1], I discuss the strong n-generator property for a Prüfer domain R defined by a conductor square of type $(\boxtimes)$, and I examine the particular case when the domain $D = \mathbb{Z}$. We find this special case of interest, as the ring $\mathrm{Int}(\mathbb{Z}) = \{g \in \mathbb{Q}[X] : g(\mathbb{Z}) \subseteq \mathbb{Z}\}$ of all integer-valued polynomials has been well-studied and is known to have the 2-generator property [7], but not the strong 2-generator property [8]. (It is not known whether there exists a non-principal strongly 2-generated ideal of $\mathrm{Int}(\mathbb{Z})$.) As the conductor of $\mathbb{Q}[X]$ into $\mathrm{Int}(\mathbb{Z})$ is zero, we cannot apply the results in [1]. We note, however, that $\mathrm{Int}(\mathbb{Z})$ is a Prüfer domain between $\mathbb{Z}[X]$ and $\mathrm{Int}(E, \mathbb{Z})$ for all finite subsets $E \subset \mathbb{Z}$; in fact, $\mathrm{Int}(\mathbb{Z}) = \bigcap\limits_{n=1}^{\infty} \mathrm{Int}(E_n, \mathbb{Z})$ where $E_n = \{1, 2, ..., n\}$. If S is any Prüfer domain between $D[X]$ and $K[X]$, is S an intersection of rings R_i such that each R_i is defined by a conductor square of type $(\boxtimes)$? In [4], Chapman, Loper, and Smith ask for an integral domain of the form $\mathrm{Int}(E, D)$ with the strong 2-generator property, such that the cardinality of E infinite and the Jacobson radical of D is zero. Can such a ring be constructed by means of the pullback of a conductor square?

5. Let D be any integral domain with quotient field K. We say that D is *atomic* if each of its elements can be written as a finite product of irreducibles of D. In [6] it is shown that the ring $\mathrm{Int}(E, D)$, where $E = \{e_1, ..., e_r\}$, is not atomic. Can one prove a similar result for a more general ring R defined by a conductor square of type $(\boxtimes)$? Using the notation of [6], let $\mathcal{A}(D)$ be the set of all irreducible elements of D, let $\mathcal{N}(D)$ be the set of all elements of D that do not admit a finite factorization length in terms of irreducibles, and let $\mathcal{F}(D)$ be the set of all elements of D that do admit a finite factorization length in terms of irreducibles. The *restricted elasticity* is defined as $\rho_{\mathfrak{r}}(D) = \sup\{\frac{m}{n} : \exists x \in \mathcal{F}(D)$ and $x_1, ..., x_m, y_1, ...y_n \in \mathcal{A}(D)$ with $x_1 \cdot ... \cdot x_m = x = y_1 \cdot ... \cdot y_n\}$. It is shown in [6] that $\rho_{\mathfrak{r}}(\mathrm{Int}(E, D)) = \infty$ and, under certain conditions, that $\mathcal{N}(D)$ is contained in the ideal $(x - e_1) \cdot ... \cdot (x - e_r)K[X] \leq \mathrm{Int}(E, D)$. It is natural for one to conjecture that the same results hold for a ring R defined by a conductor square of type $(\boxtimes)$. That is, is it true that $\rho_{\mathfrak{r}}(R) = \infty$ and that $\mathcal{N}(R)$ is contained in the ideal $F_1 \cdot ... \cdot F_r K[X] \leq R$?

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