Math 446–646

Important facts about Topological Spaces, Part II

Connected sets

- Let (X, T) be a topological space. Assume that there are two non-empty open subsets A, B of X such that $X = A \cup B$ and $A \cap B = \emptyset$. Then $\{A, B\}$ is a **partition** of X.
- Notice that if $\{A, B\}$ is a partition of X, the sets A and B are both open and closed.
- A topological space (X, T) is **connected** if it does not admit any partition.
- Some characterizations of connectedness:
  (a) (X, T) is connected if and only if the only subsets of X that are both open and closed are X and $\emptyset$.
  (b) (X, T) is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.
- Let (X, T) be a topological space, let $\{A_\alpha\}_{\alpha \in I}$ be a collection of subsets of X such that each (A_α, T_{A_α}) is connected. Assume that $\cap_{\alpha \in I} A_\alpha$ is non-empty. Then $\cup_{\alpha \in I} A_\alpha$ is connected.
- Let (X, T) be a topological space, let A be a subset of X such that (A, T_A) is connected. Then any set B such that $A \subset B \subset \overline{A}$ is connected.
- Let (X, T) be connected and $f : (X, T) \to (Y, T')$ be continuous and onto. Then (Y, T') is connected. Thus, being connected is a topological property.
- If (X, T) and (Y, T') are connected, so is $(X \times Y, T \times T')$.

- **Connectedness in the real line and its applications:**
  (a) A subset of $\mathbb{R}$ is connected if and only if it is an interval.
  (b) **Intermediate Value Theorem**: Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) \neq f(b)$. Then for each number V between $f(a)$ and $f(b)$ there is a point $u \in [a, b]$ such that $f(u) = V$
  (c) **Brouwer Fixed Point Theorem in dimension 1**: Let $f : [0, 1] \to [0, 1]$ be continuous. Then there is a point $z \in [0, 1]$ such that $f(z) = z$.
  (d) **Borsuk-Ulam Theorem in dimension 1**: Let $f : S^1 \to \mathbb{R}$ be continuous. Then there is a pair of antipodal points $z, -z \in S^1$ such that $f(z) = f(-z)$.
- Let (X, T) be a topological space and let a be a point in X. Let $\{D_\alpha\}_{\alpha \in I}$ be the collection of all connected subsets of X that contain a (notice that this collection is not empty, since $\{a\}$ is a connected set that contains a). The union $C = \cup_{\alpha \in I} D_\alpha$ is the **connected component** of X that contains a. It is the "biggest" connected subset of X that contains a.
- Two connected components in X are either disjoint or identical. Thus, X can be written as a disjoint union of its connected components, $X = \cup_{i \in I} C_i$.
- The number of connected components is a topological property, *i.e.*, two homeomorphic topological spaces have the same number of connected components.

Path connected sets

- A **path** in a topological space (X, T) is a continuous function $f : [0, 1] \to X$. (X, T) is **path connected** if for every pair of points $x, y \in X$ there is a path f such that $f(0) = x$ and $f(1) = y$.
- If (X, T) is path connected, then it is connected. However, there are connected spaces that are not path connected, such as $([0, 1]^2, T_<)$.
- Let (X, T) be path connected and $f : (X, T) \to (Y, T')$ be continuous and onto. Then (Y, T') is path connected. Thus, being path connected is a topological property.

Compact sets

- Let (X, T) be a topological space. An **open covering of X** is a collection of open sets $\{A_\alpha\}_{\alpha\in I}$ such that $X \subset \cup_{\alpha\in I} A_\alpha$.
- (X, T) is **compact** if from every open covering of X we can extract a finite subcovering.
- A characterization of compactness: (X, T) is compact if and only if whenever a family $\{F_\alpha\}_{\alpha\in I}$ of closed sets has empty intersection, then there is a finite subfamily $\{F_i\}_{i=1}^N \subset \{F_\alpha\}_{\alpha\in I}$ such that $\cap_{i=1}^N F_i = \emptyset$.
- Let (X, T) be a compact topological space. If $C \subset X$ is closed, then C is compact.
- Let (X, T) be a Hausdorff topological space. If $C \subset X$ is compact, then C is closed.
- Let (X, T) be a compact space and $f : (X, T) \to (Y, T')$ be continuous and onto. Then (Y, T') is compact. Thus, being compact is a topological property.
- If (X, T) and (Y, T') are compact, so is $(X \times Y, T \times T')$.
- Let (X, T) be a compact space and let (Y, T') be a Hausdorff space. Let $f : X \to Y$ be bijective and continuous. Then f is a homeomorphism.
- Let (X, T) be a compact, Hausdorff topological space such that all its points are accumulation points. Then X has uncountably many points.
- **Compact subsets of** $\mathbb{R}^n$: A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.