Math 446–646

Important facts about Topological Spaces

- A **topology** on X is a collection T of subsets of X such that:
  (1) $\emptyset \in T$
  (2) $X \in T$
  (3) If A and B are in T, then $A \cap B \in T$.
  (4) If for each $\alpha \in I$, $A_\alpha \in T$, then $\bigcup_{\alpha \in I} A_\alpha \in T$.
- The elements of T are called **open sets**.
- A subset F of X is closed if F^c is open (*i.e.*, if $F^c \in T$).
- A subset $N \subset X$ is a **neighborhood** of a point x if there is an open set $O \in T$ such that $x \in O \subset N$.
- Let T_1 and T_2 be two topologies on the same space X. If $T_1 \subset T_2$, we say that T_1 is coarser than T_2, or that T_2 is finer than T_1.
- Some important examples:
  (1) $T_{triv} = \{X, \emptyset\}$, the **trivial topology**.
  (2) $T_{dis} = \mathcal{P}(X)$, the **discrete topology**. In the discrete topology, every subset of X is both open and closed.
  (3) If X has infinitely many elements, $T_F = \{A \subset X \mid A^c \text{ is finite or } A^c = X\}$, the **finite-complement topology**.
  (4) In $\mathbb{R}$ we consider the **lower-limit topology** T_l, whose elements are unions of intervals of the type $[a, b)$, where $a < b$.
  (5) If (X, d) is a metric space, the collection T_d of unions of open balls with respect to d is a topology, the **topology induced by the metric** d.
  In particular, the topology induced in $\mathbb{R}^n$ by the Euclidean metric is called the **usual topology** on $\mathbb{R}^n$, and denoted by T_u.
  (6) Let X be a space with an total order relation $\leq$. Assume that X does not have a smallest or a biggest element. Denote by $(a, b) = \{x \in X \mid a < x \text{ and } x < b\}$. We define the **order topology** $T_<$ on X as follows: A set O is open if it is union of sets of the type (a, b) defined above.

(7) In example (6), assume that X has a smallest element a_0 and a biggest element b_0. Denote by $[a_0, c) = \{x \in X \,|\, a_0 \leq x \text{ and } x < c\}$ and by $(c, b_0] = \{x \in X \,|\, c < x \text{ and } x \leq b_0\}$. A set O is open if it is union of sets of the types (c, d), $[a_0, c)$ $(c, b_0]$. We also call the topology $T_<$, the order topology.

(8) Let (X, T) be a topological space, let $Y \subset X$. Then Y inherits a topology from that of X, called T_Y, the **relative topology in** Y, as follows: A set $O \subset Y$ belongs to T_Y if and only there exists an open set $U \in T$ such that $O = U \cap Y$.

Note that it is possible for a set O to be open in the topology T_Y and not to be open in the topology T.

(9) Let $(X_1, T_1), \ldots, (X_n, T_n)$ be n topological spaces. We define a topology on $\prod_{i=1}^n X_i$, called the **product topology** T_{prod} as follows: A set $O \in T_{prod}$ if and only if O is a union of sets of the form $O_1 \times \cdots \times O_n$, where $O_i \in T_i$ for each $i = 1, \ldots, n$.

(10) **Identification topologies**: See section 3.8 in the book.

- A collection $\mathcal{B} \subset T$ is a **basis** of the topology T if every open set in T is a union of elements of $\mathcal{B}$. A basis is useful to describe a topological space without having to specify all the open sets.
- Bases make it very easy to compare two topologies (instead of comparing all the open sets, we only need check the elements of the basis). Given a basis $\mathcal{B}_1$ of the topology T_1 and a basis $\mathcal{B}_2$ of the topology T_2, $T_1 \subset T_2$ if and only if for every element $B \in \mathcal{B}_1$ and every point $x \in B$ there exists an element $\widetilde{B} \in \mathcal{B}_2$ such that $x \in \widetilde{B} \subset B$.

- Let (X, T) be a topological space and let $A \subset X$. A point $x \in X$ is in the **closure** of A if for every open set O containing x, $O \cap A \neq \emptyset$.
- The closure of A is denoted by $\overline{A}$. $\overline{A} = \bigcap F$, where the intersection is taken over all the closed sets F that contain A (thus, $\overline{A}$ is the "smallest" closed set that contains A). Therefore, A is closed if and only if $A = \overline{A}$.
- The **interior** of A is the union of all the open sets contained in A. It is denoted by A^o. (Thus, A^o is the "biggest" open set contained in A). A is open if and only if $A = A^o$.
- The **boundary** of A is $\text{Bdry}(A) = \overline{A} \cap \overline{A^c}$. A point x is in the boundary of A if and only if for every open set O containing x, $O \cap A \neq \emptyset$ and $O \cap A^c \neq \emptyset$.
- Review the properties of interior and closure seen in section 3.6, in the homework assignment for 3.6 and in class.
- A function $f : (X, T) \to (Y, T')$ is **continuous** if and only if for every $O \in T'$, we have $f^{-1}(O) \in T$ (*i.e.*, the inverse image of any open set is open). Equivalently, f is continuous if and only if for every closed set F in Y, $f^{-1}(F)$ is closed in X.
- Some important continuous functions:
  (1) If f is a constant function, then f is continuous.
  (2) If $f : (X, T) \to (Y, T')$ is continuous and $g : (Y, T') \to (Z, T'')$ is continuous, then the composition $g \circ f : (X, T) \to (Z, T'')$ is continuous.
  (3) The identity function $Id : (X, T) \to (X, T)$ is continuous. However, if we consider two different topologies in X, the identity function $Id : (X, T) \to (X, T')$ may not be continuous. Example: $Id : (\mathbb{R}, T_u) \to (\mathbb{R}, T_l)$ is not continuous.
  (4) The projection function $p_k : (\prod_{i=1}^n X_i, T_{prod}) \to (X_k, T_k)$ is continuous.
  (5) Let (X, T) be a topological space. If $Y \subset X$, then the inclusion function $i : (Y, T_Y) \to (X, T)$ (defined as $i(y) = y$) is continuous.
  (6) Let $f : (X, T) \to (Y, T')$ be continuous. Let $Z \subset X$. Then the restriction function $f|_Z : (Z, T_Z) \to (Y, T')$ is continuous.

(7) **The pasting lemma**: Let (X, T) and (Y, T') be topological spaces, and assume that $X = A \cup B$, where both A and B are closed sets. Let $f : (A, T_A) \to (Y, T')$ and $g : (B, T_B) \to (Y, T')$ be continuous functions such that, for every $x \in A \cap B$, $f(x) = g(x)$. Then the function $h : (X, T) \to (Y, T')$, defined as

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

- A function $f : (X, T) \to (Y, T')$ is **open** if and only if for every $O \in T$, we have $f(O) \in T'$ (*i.e.*, the image of any open set is open).
- An function may be open but not continuous. Also, a function may be continuous but not open.
- A **homeomorphism** is a function $f : (X, T) \to (Y, T')$ that is bijective, continuous and has a continuous inverse. If there is a homeomorphism between two spaces (X, T) and (Y, T'), we say that the two spaces are homeomorphic.
- If f is bijective, continuous and open, then f is a homeomorphism.
- We say that a property of a space (X, T) is a **topological property** if every space that is homeomorphic to (X, T) also has that property.
- Topological properties will help us decide if two different topological spaces are homeomorphic or not.
- Some topological properties:

(a) A space (X, T) is **metrizable** if there is a metric d such that $T_d = T$.
Example: (X, T_{dis}) is always metrizable, with the metric

$$d(a, b) = \begin{cases} 0 & a = b \\ 1 & a \neq b \end{cases}$$

(b) **Property T_1:** A topological space (X, T) has property T_1 if for any point $x \in X$, the set $\{x\}$ is closed.

(c) **Hausdorff property** or **property** T_2: A topological space (X, T) has the Hausdorff property if given two different point $x, y \in X$, there exist open sets $O_1, O_2 \in T$ such that $x \in O_1$, $y \in O_2$ and $O_1 \cap O_2 = \emptyset$.

Any space with the property T_2 also has property T_1, but the converse is not true.

If (X, T) is metrizable, then it has the Hausdorff property.

Example: In $\mathbb{N}$ we consider the topology T formed by $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{k, k+1, \ldots\}$. Then $(\mathbb{N}, T)$ is not metrizable because it does not have the Hausdorff property.

(d) The **first-countability axiom**: (X, T) satisfies the first-countability axiom if X has a countable basis for its topology at each one of its points, *i.e.* at each point $x \in X$ there is a countable collection $\mathcal{B}_x$ of open sets containing x, such that for any neighborhood N of x, there is an element $B \in \mathcal{B}_x$ such that $x \in B \subset N$.

If $X, T)$ is metrizable, then it satisfies the first-countability axiom.

(e) The **second-countability axiom**: (X, T) satisfies the second-countability axiom if X has a countable basis for its topology.

A space that satisfies the second-countability axiom also satisfies the first-countability axiom.

Not every metrizable space satisfies the second-countability axiom.

(f) A space (X, T) is **separable** if there is a countable subset $S \in X$ such that $\overline{S} = X$.

For example, $(\mathbb{R}, T_u)$ is separable, because $\overline{\mathbb{Q}} = \mathbb{R}$.